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Selected contributions at the 3rd International
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June 24–28, 2013, Moscow, Russia

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Теоретическая физика и её новые приложения

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Preface

This volume is based on the selected contributions
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Предисловие

Этот том состоит из избранных докладов, сделанных
на 3-й Международной конференции по теоретической физике,
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A topos theory for quantum mechanics

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Abstract: *Any quantum system that has an algebra of physical attributes represented by a *-algebra, $\mathcal{A}$, on a Hilbert space that carries a unitary representation of its symmetry group has its own real number system for the values of its attributes. They are called quantum real numbers (qr-numbers). When its state space $\mathcal{E}_{\mathcal{S}}(\mathcal{A})$ has the weak topology generated by the real - valued functions $a_Q : \mathcal{E}_{\mathcal{S}}(\mathcal{A}) \to \mathbb{R}$ given by $a(\rho) = Tr(\rho.\hat{A}) \ : \forall \rho \ \in \mathcal{E}_{\mathcal{S}}(\mathcal{A})$ and $\hat{A} \in \mathcal{A}$, the qr-numbers are sections of $\mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$, the sheaf of Dedekind real numbers in the spatial topos $Shv(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$. The open subsets of $\mathcal{E}_{\mathcal{S}}(\mathcal{A})$ are the conditions of the system, the internal logic is intuitionistic.*
The standard real number value of a physical attribute obtained in a measurement is a constant qr-number approximations to the attribute's actual qr-number value.
Each quantum particle with positive mass moves in a spatial continuum that is isomorphic to $\mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))^3$. This continuum is sufficiently non-classical that a single particle can have a quantum trajectory which passes through two classically separated slits and the two particles in the Bohm-Bell experiment stay close to each other so that Einstein locality is retained in quantum space.

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The incompleteness of quantum theory

In a letter to Schrödinger dated December 22 1950 [20] in talking about quantum theory, Einstein remarked that,

> If one wants to consider the quantum theory as final (in principle), then one must believe that a more complete description would be useless because there would be no laws for it. If that were so then physics could only claim the interest of shopkeepers and engineers, the whole thing would be a wretched bungle.

The following is an attempt to make quantum theory complete in the sense of Einstein [7] (1933), "A complete system of theoretical physics is made up of concepts, fundamental laws which are supposed to be valid for those concepts and conclusions to be reached by logical deduction. It is these conclusions which must correspond with our separate experiences;'. Some of the experiences of quantum-physics, mainly those associated with measurements, that reveal lacunae in the standard quantum theory will be discussed.

Other topos theoretical approaches

Different topos theoretical formulations of quantum theory has been developed, for a recent survey see Flori [8].Their aims are similar to those of the quantum real numbers approach but the emphases are quite different. All start from algebraic formulations of quantum theory; Doering and Isham [6]use von Neumann algebras, Heunen and Spitters [11] use C^*-algebras, and we use O^*-algebras. The first two take a non-commutative algebra and cover it by a family of commutative sub-algebras that are objects in a topos of presheaves. We call this the commutative sub-algebra approach.

What is an empirical fact? If physical quantities can be measured with infinite accuracy then only commutative sub-algebras can be measured because of Heisenberg's uncertainty principle. This is the understanding that seems to be accepted in the commutative sub-algebras approach. The qr-numbers approach accepts that

empirical numerical facts are only ever known with finite accuracy. In this view, Heisenberg's uncertainty relations puts limits on the level of accuracy attainable in measuring non-commuting elements but does not prohibit their measurement. Furthermore the commutative sub-algebra approach seems to accept that physical qualities only have numerical values when they are observed. In the qr-number approach they have numerical values even when not being observed.

All seek to revise the mathematical structure of quantum theory in order that it can be re-interpreted in realist terms[1] rather than as an instrumentalist theory.

Outline of the paper

- The qr-number hypothesis.
- Mathematical structures
- Quantum space and Heisenberg's Inequalities.
- Aspects of measurement
- Heisenberg's Inequalities.
- Evidence that the spatial continuum of quantum phenomena is not classical .
- Entanglement in the qr-numbers model: EPR-Bohm-Bell experiment.
- The double slit experiment.
- A deterministic violation of Bell's theorem.
- qr-number equations of motion.
- Summing up.

The talk is based on work with Thomas Durt and the late Murray Adelman. I wish to thank Prof Timur Kamalov for his invitation to talk at this conference.

The quantum real number hypothesis:

The paradigm shift from classical to quantum theory occurs through the change in the type of real numbers that physical variables can take as quantitative values.

The real numbers for a quantum system are Dedekind real numbers in a spatial topos built upon the system's state space whose open sets are the domains of definition of the numbers.

The "direct connection between observation properties and properties possessed by the independently existing object" [4] is cut, an indirect connection is made through measurement processes. Each experimental measurement has a limited level of accuracy, within which the measured attribute's qr-number value is approximated by a standard real number.

In the qr-number model:

- the Lüders-von Neumann transformation (collapse hypothesis) appears as an approximation to the change in the qr-number values in a measurement of the first kind,
- The Born rule gives a standard real number approximation to the qr-number probability,
- Heisenberg's uncertainty relations restricts the accuracy at which some attributes can be measured simultaneously but doesn't prohibit their measurement,

[1] A realist theory is one in which the systems and their attributes exist and take values independently of whether they are measured.

- massive particles have trajectories in the qr-number space to extents that allow (1) a single particle to pass through a double slit without being detected at either slit and (2) two entangled particles to stay close to each other,
- the qr-number values $(\vec{q}_Q(U)(t), \vec{p}_Q(U)(t))$ of a massive particle satisfy Hamilton's equations of motion that can be approximated by Heisenberg's operator equations averaged over U for certain families of open sets.

Mathematical structures

A quantum system with a Hilbert space $\mathcal{H}$ carrying a unitary representation U of its symmetry group G has physical attributes (identifiable qualities) which are represented by e.s.a. operators on dense domains of $\mathcal{H}$ that:

- form an O*-algebra $\mathcal{A}$; e.g., the representation dU of the enveloping algebra $\mathcal{E}(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ of G,
- have a common domain $\mathcal{D}^\infty(U)$, the set of C^∞-vectors for the representation U,
- are operationally identifiable through their transformations under subgroups of G. e.g., Euclidean group transformations identify the position attributes.

The state space $\mathcal{E}_\mathcal{S}(\mathcal{A})$ is the space of normalized strongly positive linear functionals on $\mathcal{A}$.

- The states are normalized trace class operators: convex combinations of projections $\mathcal{P} = |\phi\rangle\langle\phi|$ onto unit vectors $\phi \in \mathcal{D}^\infty(U)$.
- $\mathcal{E}_\mathcal{S}(\mathcal{A})$ has the weak topology generated by the functions $a : \mathcal{E}_\mathcal{S}(\mathcal{A}) \to \mathbb{R}$ given by $a(\rho) = Tr(\rho\hat{A}) \ : \forall \rho \in \mathcal{E}_\mathcal{S}(\mathcal{A})$ and labeled by e.s.a. operators $\hat{A} \in \mathcal{A}$.
- A typical open setis a finite intersection $\mathcal{N}(\rho_0; \hat{A}; \epsilon) = \{\rho;\ |Tr\rho\hat{A} - Tr\rho_0\hat{A}| < \epsilon\}$, for $\hat{A} \in \mathcal{A}$, $\rho_0 \in \mathcal{E}_\mathcal{S}(\mathcal{A})$, $\epsilon > 0$. When $\mathcal{A} = dU(\mathcal{E}(\mathcal{G}))$ any open set is a union of $\nu(\rho_k; \delta) = \{\rho | Tr|\rho_k - \rho| < \delta\}$ for $\rho_k \in \mathcal{E}_\mathcal{S}(\mathcal{A})$ and $\delta > 0$.

Such a quantum system will be labelled $\{\mathcal{H}, \mathcal{A}, \mathcal{E}_\mathcal{S}(\mathcal{A})\}$.

qr-numbers

qr-numbers depend upon the quantum system as they are sections of the sheaf of Dedekind reals $\mathbb{R}_\mathrm{D}(\mathcal{E}_\mathcal{S}(\mathcal{A}))$ in $\mathrm{Shv}(\mathcal{E}_\mathcal{S}(\mathcal{A}))$, the topos of sheaves on the topological space $\mathcal{E}_\mathcal{S}(\mathcal{A})$.

$\mathbb{R}_\mathrm{D}(\mathcal{E}_\mathcal{S}(\mathcal{A}))$ is isomorphic to $\mathcal{C}(\mathcal{E}_\mathcal{S}(\mathcal{A}))$, the sheaf of germs of continuous real-valued functions on $\mathcal{E}_\mathcal{S}(\mathcal{A})$. Two functions have the same germ at $\rho \in \mathcal{E}_\mathcal{S}(\mathcal{A})$ if they agree on some open neighbourhood of ρ.

For each non-empty $U \in \mathcal{O}(\mathcal{E}_\mathcal{S}(\mathcal{A}))$, the subsheaf $\mathbb{R}_\mathrm{D}(U)$

- has integers $\mathbb{Z}(U)$, rationals $\mathbb{Q}(U)$ and Cauchy reals $\mathbb{R}_\mathrm{C}(U)$ as subsheaves of locally constant functions;
- has orders $<$ and $\leq$ compatible with those on the rationals $\mathbb{Q}(U)$ but the inequality $<$ is not total because trichotomy, $x > 0 \vee x = 0 \vee x < 0$, is not satisfied. Moreover $\leq$ is not equivalent to $< \vee =$;
- is closed under the commutative, associative, distributive binary operations $+$ and $\times$, has $0 \neq 1$ and is a residue field, i.e., if $b \in \mathbb{R}_\mathrm{D}(U)$ is not invertible then $b = 0$;
- has a distance function $|\cdot|$ which defines a metric with respect to which it is a complete metric space in which $\mathbb{Q}(U)$ dense. A section $b \in \mathbb{R}_\mathrm{D}(U)$ is apart from 0 iff $|b| > 0$. $\mathbb{R}_\mathrm{D}(U)$ is an apartness field, i.e., $\forall b \in \mathbb{R}_\mathrm{D}(U)$, $|b| > 0$ iff b is invertible.
- are not Dedekind complete, as least upper bounds need not exist, and they are not Archimedean, because there are infinitesimal qr-numbers. For example, the expectation value $a(\rho_0) = Tr\rho_0\hat{A}$ for a given state and a given operator is an infinitesimal qr-number because there is no open set $V \neq \emptyset$ such that $a(\rho_0)|_V > 0 \vee a(\rho_0)|_V < 0$.

Locally linear qr-numbers $\mathbb{A}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$

They are important in the construction of qr-numbers and form a subsheaf of $\mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$. Given an essentially self-adjoint $\hat{A} \in \mathcal{A}$ there is a locally linear real-valued function a on $\mathcal{E}_{\mathcal{S}}(\mathcal{A})$ with $a(\rho) = Tr\rho\hat{A}$ for $\rho \in \mathcal{E}_{\mathcal{S}}(\mathcal{A})$.

- $\mathbb{A}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$ is dense in $\mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$.
- Every qr-number is a continuous real function of locally linear qr-numbers.
- If $W \neq \emptyset$, $a(W) = b(W)$ if and only if the defining operators are equal, $\hat{A} = \hat{B}$.
- If $W \neq V$ are open sets it is possible that the range of $a(W)$ is the same as that of $a(V)$.
- The prolongation by zero [21] of a locally linear qr-number $a(W)$, for $W \neq \emptyset$, gives a globally defined extended locally linear qr-number.

qr-numbers generated by a single operator $\hat{A}$ generates sub-sheaves $\mathbb{A}^{\hat{A}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$ of $\mathbb{A}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$ and ${\mathbb{R}_{\mathrm{D}}}^{\hat{A}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$ of $\mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$, both contain the integers $\mathbb{Z}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$, rationals $\mathbb{Q}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$ as sub-sheaves of locally constant functions. The order relations and the distance function defined on $\mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$ restrict to ${\mathbb{R}_{\mathrm{D}}}^{\hat{A}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$ so that $\mathbb{Q}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$ is dense in ${\mathbb{R}_{\mathrm{D}}}^{\hat{A}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$. If the numerical range of $\hat{A}$ is $\mathbb{R}$ then ${\mathbb{R}_{\mathrm{D}}}^{\hat{A}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$ is a complete metric space.

qr-number geometry

In Riemann's 1854 lecture *On the hypotheses which lie at the foundations of geometry* there are two hypotheses :

- A topological hypothesis: locations are fixed by allocating multiplets of real numbers.
- A metrical hypothesis: the distance between the located points is given by a metric function leading to Riemannian geometries.

Riemann did not explore the choices of real numbers and this has been mostly ignored since. On the other hand Riemann saw that there was a choice of the metric and that choice would be determined by the physics.

We propose that the appropriate real numbers to fix the locations of a single massive quantum particle with symmetry group G are the qr-numbers $\mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$ with $\mathcal{A} = dU(\mathcal{E}(\mathcal{G}))$),

- its quantum space is topologically given as $\mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))^3$.
- the cartesian coordinate axes are parametrized by $\mathbb{A}^{x^j_Q}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$ generated by the position operators $\hat{X}^j$, for $j = 1, 2, 3$.
- The triplet $\vec{x}_Q(W)$ for non-empty $W \in \mathcal{O}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$ is an open set in $\mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))$ by construction.
- The graph of $\vec{x}_Q(W)$ is $(W, O_{\vec{x}})$ with $O_{\vec{x}}$ an open subset of the Euclidean space $\mathbb{R}^3$.

For a system of N massive quantum particle the locations are fixed by the real numbers $\mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{A}))^N$.

Aspects of Measurement

Every measurement is obtained at a finite level of precision [16]. Every experiment has a confidence level, $0 \leq (1 - \epsilon) \leq 1$ as well as a measure of precision $\kappa > 0$, when the confidence level is unity, when $\epsilon = 0$, the experimenter is has complete confidence in the experimental results, including the measure of precision.

The relation between the qr-number value of an attribute and the standard rational number obtained in an experimental measurement is elucidated through the process of ϵ sharp collimation [3] in a single slit.

The ϵ sharp collimation of $\hat{A}$ in I_a when the system has the condition W is sufficient to give a standard real approximation to its qr-number value. For an interval I_a, midpoint a_0 and width $|I_a|$, if W is such that $a(W) \in I_a$ and $(a^2(W) - a(W)^2) \leq \frac{\epsilon}{4}|I_a|^2$ then $a(W) = a_0 1(W)$ with precision $|I_a|/2$.

Strictly ϵ sharp collimation is a stronger version of ϵ sharp collimation. Let $\hat{P}^{\hat{A}}(I_a)$ be the spectral projection operator of $\hat{A}$ on I_a, then $\hat{A}$ is strictly ϵ sharp collimated in I_a on W if it is ϵ sharp collimated on W and for all $\rho \in W$ $Tr|\rho - \hat{P}^{\hat{A}}(I_a)\rho\hat{P}^{\hat{A}}(I_a)| < \epsilon$.

Theorem 1. *If $a_0 \in \sigma(\hat{A})$ and $\rho_0 = |\psi_0\rangle\langle\psi_0|$ is an eigenstate of $\hat{A}$ at a_0 with $\hat{P}^{\hat{A}}(I_a)\rho_0\hat{P}^{\hat{A}}(I_a) = \rho_0$, then $\forall\epsilon > 0, \exists\delta > 0$ such that $\hat{A}$ is strictly ϵ sharp collimated in I_a on $\nu(\rho_0, \delta)$.*

The spectral projection operator $\hat{P}^{\hat{A}}(I_a)$ has the qr-number value $\pi^{\hat{A}}(I_a)(W)$ in the condition W. $\hat{A}$ is ϵ sharp located in the interval I_a in the condition W when $(1-\epsilon) < \pi^{\hat{A}}(I_a)(W) \leq 1$, this is a stronger version of weak location which means only that the range of $a(W) \subseteq I_a$.

Theorem 2. *If $\hat{A}$ is ϵ sharp collimated in I_a on W, then $\hat{A}$ is ϵ sharp located in I_a on W.*

Lüders-von Neumann Collapse is a standard way of understanding quantum measurement. The collapse hypothesis states that in a measurement of $\hat{A}$ an initial state ρ_0 "collapses" to the state $\rho_0' = \frac{\hat{P}^A(I_a)\rho_0\hat{P}^A(I_a)}{Tr(\hat{P}^A(I_a)\rho_0)}$. The following theorem reveals the sense in which the collapse postulate gives a good approximation to the qr-number value of the observable.

Theorem 3. *If the system is prepared in $U = \nu(\rho_0, \delta)$ and then $\hat{A}$ is strictly ϵ sharp collimated in I_a on W, then any $\hat{B} \in \mathcal{A}$ will have the qr-number value $b(U \cap W) \approx (Tr\rho_0'\hat{B})1(U \cap W)$ to accuracy proportional to $\delta + 2\epsilon$ with constant of proportionality depending on $\hat{B}$.*

When $\hat{A}$ is strictly ϵ sharply collimated in I_a the qr-number value of any attribute changes as if it had undergone a transformation $\hat{B} \to \frac{\hat{P}\cdot\hat{B}\cdot\hat{P}}{Tr(\hat{P}\cdot\hat{\rho}_0)}$.

Quantum mechanical probabilities

The spectral families of self adjoint operators are used [3] to define quantum probability measures on $\mathbb{R}$. If $\hat{P}^{\hat{A}}(S)$ is the spectral projection operator of $\hat{A}$ on the Borel subset S of $\mathbb{R}$, then in the standard interpretation $\mu_\rho^{\hat{A}}(S) = Tr\rho\hat{P}^{\hat{A}}(S)$ is the probability that when the system is in the state ρ a measurement of $\hat{A}$ gives a result in the set S.

The qr-number probability that $a(U)$ lies in S is $\mu_U^{\hat{A}}(S) = \pi^{\hat{A}}(S)(U)$, the qr-number value of $\hat{P}^{\hat{A}}(S)$ at U.

If $U = \nu(\rho_s; \delta)$ for $\delta \ll 1$ then, for all Borel sets S, $\pi^{\hat{A}}(S)(U) \approx Tr\rho_s\hat{P}^{\hat{A}}(S) = \mu_{\rho_s}^{\hat{A}}(S)$, the standard quantum mechanical probability when the system is in the state ρ_s. so that $|\mu_U^{\hat{A}}(S) - \mu_{\rho_s}^{\hat{A}}(S)| < \delta$.

In the operational approach of Busch et al. [3] effects are positive operators that satisfy $\hat{0} \leq \hat{B} \leq \hat{I}$. They form a convex subset of the bounded operators $\mathcal{B}(\mathcal{H})$. When quantum mechanical probabiities are described by POV (positive operator valued) measures [3], the effects lie in their range and are defined as the observables. If $\mathcal{F}$ is a σ- algebra of subset of a non-empty set M then a normalised positive operator valued (POV) measure is map $E : \mathcal{F} \to \mathcal{B}(\mathcal{H})$ that satisfies: (i) $E(S) \geq \hat{0}$ for all $S \in \mathcal{F}$, (ii) $E(M) = \hat{I}$ and (iii) $E(\cup S_k) = \sum E(S_k)$ for sequences $\{S_k\} \in \mathcal{F}$ with $S_j \cap S_i = \emptyset$ if $i \neq j$, convergence being in the weak operator topology on $\mathcal{B}(\mathcal{H})$.

In this operational approach it is the properties that are unsharp, the numerical results of a measurement are still assumed to be sharp. In contrast, in the qr-number approach, it is the measured values of the properties that are unsharp in the sense of not being infinitely precise..

On the other hand, the effects also have qr-number values, the qr-number value of the effect $E(S)$ when the system is in the condition W is $e_S(W)$. Then a measurement of $E(S)$ is ϵ sharp collimated in $]a, b[\subset]0, 1[$ on the condition W if the range of $e_S(W)$ lies in $]a, b[$ and $(e_S^2(W) - e_S(W)^2) \leq \epsilon\frac{(b-a)^2}{4}$.

Statistics: ontological and epistemological conditions

When a system is experimentally prepared in an open set W of state space, then its epistemological condition is W. The ontological condition of the system can be any non-empty open set $V \subset W$ because the values of qualities defined to extent V will satisfy the experimental restrictions imposed on qualities defined to extent W.

Usually the epistemic condition is determined for an ensemble of systems, each member of the ensemble has an ontic condition $V \subset W$. This leads to an ignorance interpretation for the statistics because attributes can have different qr-number values on different V.

We have shown [10] that if the epistemological condition for an ensemble of systems is the open set $\nu(\rho_0, \delta)$ for $0 < \delta \ll 1$ then the outcomes of a dichotomic experiment are well approximated by expectation values calculated at the quantum state ρ_0 from which Born's quantum probability rule can be obtained.

When the epistemological condition W ensures that $\hat{A}$ is strictly ϵ sharply collimated in I_a then any ontic condition $U_k \cap W \neq \emptyset$ registers a distinct value $a_k \in I_a$ for $\hat{A}$ to accuracy ϵ. The measurement problem arises because quantum mechanics predicts only a probability distribution of the values obtained by measuring a physical quantity on an ensemble of systems which are all prepared identically. But probabilities can only be determined if each outcome can be observationally distinct. This is obtained in the qr-number description but not in the standard model [9].

Heisenberg Inequalities

In the qr-number model, they give limitations on the precision with which the qr-number values of two attributes represented by non-commuting operators can be simultaneously approximated by standard real number values on an open set W. Let $\hat{A},\ \hat{B} \in dU(\mathcal{E}(\mathcal{G}))$ be essentially self adjoint on $\mathcal{D}^\infty(U) \subset \mathcal{H}$.

Theorem 4. *If $\hat{A}$ is ϵ-sharp collimated in I_a and $\hat{B}$ is ϵ-sharp collimated in I_b when the system is the condition W and $\imath\hat{C} = [\hat{A},\ \hat{B}]$ then*

$$|I_a||I_b| \geq 2|c(W)|/\epsilon \tag{1}$$

When precision is given by the half-width of the interval, the Heisenberg's uncertainty principle limits the precision with which the attributes can be realised simultaneously in the condition W.

Corollary 1. *Let $\hat{Q}$ and $\hat{P}$ represent the position and its conjugate momentum of a massive particle, i.e., $\imath[\hat{P},\ \hat{Q}] = \hbar$, and let I_q and I_p be slits for the conjugate variables. If a particle in a condition W is ϵ-sharp collimated through both slits then the product of the widths of the slits must satisfy,*

$$|I_q||I_p| \geq\ 2\hbar/\epsilon \tag{2}$$

This result determines the minimum area in the classical phase space that is required if a particle is to be ϵ-sharp collimated in both the $\hat{Q}$ and $\hat{P}$ attributes.

Entangled two-particle conditions

Let $\Psi(1,2) = \frac{1}{\sqrt{2}}(\phi_R(1)\phi_L(2) \pm \phi_L(1)\phi_R(2))$ with orthogonal single particle wave functions, $\{\phi_R(j), \phi_L(j)\}$ which span 2-dimensional subspaces $\hat{P}(j)\mathcal{H}(j),\ j = 1, 2$. The entangled pure state is $\rho_0(1,2) = \hat{P}_{\Psi(1,2)}$ whose reduced states are mixed states $\rho_0(j) = \frac{1}{2}\hat{P}(j)$ where the projection operator $\hat{P}(j) = (\hat{P}_{\hat{U}\phi_R(j)} + \hat{P}_{\hat{U}\phi_L(j)})$ for any isometry $\hat{U}$ on $\hat{P}(j)\mathcal{H}(j)$.

The two particle condition $W_0(1,2) = \nu(\hat{P}_{\Psi(1,2)}; \epsilon)$ reduces to the single particle conditions $\tilde{W}_0(j) = \nu(\rho_0(j); \delta_j),\ j = 1, 2$, for $\delta_j \leq \epsilon$.

If $\epsilon < \frac{1}{2}$ then $W_0(1,2)$ is an entangled condition as it contains no product states and there are no pure states in $\tilde{W}_0(j) = \nu(\rho_0(j); \delta_j)$. Therefore the measurement of any single particle attribute $\hat{A}(j)$ in the condition $\tilde{W}_0(j)$ cannot be measured as a standard real number.

Because $\nu(\rho_0(j), \epsilon) \approx \frac{1}{2}\nu(P_{\hat{U}\phi_R(j)}; \epsilon) + \frac{1}{2}\nu(P_{\hat{U}\phi_L(j)}; \epsilon)$, where the conditions $\nu(\hat{P}_{\hat{U}\phi_K(j)}; \epsilon), K = L, R$, contain pure states and hence support the measurement of some attributes of the individual particle. Each attribute $\hat{A}(j)$ has qr-number value, if $\epsilon \ll 1$,

$$a(\tilde{W}_0(j)) \approx \frac{1}{2}(a(\nu(\hat{P}_{\hat{U}\phi_R(j)}; \epsilon)) + \frac{1}{2}(a(\nu(\hat{P}_{\hat{U}\phi_L(j)}; \epsilon)) \tag{3}$$

which can be used in an experiment to separate the possible outcomes.

Evidence that the spatial continuum of quantum phenomena is not classical.

Bell's experiments

(a) In the Einstein-Podolsky-Rosen-Bohm-Bell experiments for two spin one-half massive particles, the particles are prepared so that the sums of their momenta and their spins are both zero. They are sent to two Stern-Gerlach apparatuses, $B_R,\ B_L$, a large distance apart whose magnetic fields can be set independently in directions $\vec{u}(R),\ \vec{u}(L)$.

If quantum space is assumed to be classical and each particle is assumed to arrive at one of $B_R,\ B_L$ then the experiment contravenes Einstein locality because changing the direction of the magnetic field in one apparatus affects the particle in the other.

The qr-numbers approach maintains Einstein locality in the quantum space of the entangled two particle system because the single particles are always close to each other in their qr-number space and hence can always interact. [13]

It gives the usual quantum mechanical results for the experiment except when the particles have different masses because the detected mass is the average of the masses of the particles.

In a suitably prepared condition the qr-number trajectory for each particle goes both to B_L and up along $\vec{u}_L$ and to B_R and down along $\vec{u}_R$ to extent $\tilde{W}_L^+ \cup \tilde{W}_R^-$, and each goes both to B_L and down along $\vec{u}_L$ and to B_R and up along $\vec{u}_R$ to extent $\tilde{W}_L^- \cup \tilde{W}_R^+$. For example a particle with ontic condition V_n registers in a detector in the upper sector of B_R only if $V_n \cap \tilde{W}_R^+ \neq \emptyset$.

The double slit experiment

The building up process is described deterministically in the qr-number model of quantum mechanics [14].

A quantum particle can simultaneously pass through two slits $I_z^+,\ I_z^-$ that are separated in classical space $I_z^+ \cap I_z^- = \emptyset$ if its location in qr-number space is an open set $z_Q(V_+ \cup V_-)$ with $z_Q(V_+) \subset I_z^+$ and $z_Q(V_-) \subset I_z^-$ so that $\hat{Z}$ is not ϵ located in I_z^+ on V_+ nor in I_z^- on $V-$.

The particle has a qr-number trajectory that passes through the double slits and arrives at a single classical location on the detector screen. If the de Broglie relation $p_y = \frac{h}{\lambda_y}$ between the momentum p_y and a wavelength λ_y is assumed then the qr-number path difference between the sub-paths is approximately λ_y when the qr-number trajectory ends in the vicinity of the first maxima of the interference pattern.

When sufficiently many particles have been prepared that the union of their ontological conditions covers $W_m(\epsilon)$, the qr-number probability that a particle, prepared in the epistemic condition $W_m(\epsilon)$, is detected in the d^{th} slit is given by the qr-number $\pi_d(t_2)(W_m(\epsilon)) = \pi_d(t_2)(\cup_\alpha V_\alpha)$, the qr-number value of the projection operator $\hat{P}^{\hat{Z}(t_2)}(I_d)$ in the condition $W_m(\epsilon)$. The standard formula for the interference pattern is obtained when $W_m(\epsilon) = \nu(\rho_m, \epsilon)$ with $\rho_m = |\psi_m\rangle\langle\psi_m|$ for $\psi_m = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-)$.

A deterministic violation of Bell's theorem

Consider the Bell-Bohm experiment as a measurement of the first kind of the two particle attribute $\hat{C}(\vec{u}_L, \vec{u}_R) = \sigma_1.\vec{u}_L \otimes \sigma_2.\vec{u}_R$ which is repeated for pairs of particles prepared in the epistemic condition $W_0(\epsilon) = \nu(\rho_0; \epsilon)$.

The qr-number value of the attribute $\hat{C}(\vec{u}_L, \vec{u}_R)$ for the n^{th} pair in the condition $V_n(1,2) \subset W_0(\epsilon)$ is $c(\vec{u}_L, \vec{u}_R)_Q(V_n(1,2))$. Now $\hat{C}(\vec{u}_L, \vec{u}_R)$ is a symmetry with $\|\hat{C}(\vec{u}_L, \vec{u}_R)\| = 1$. There exists a projection operator $\hat{E}(\vec{u}_L, \vec{u}_R)$ such that $2\hat{E}(\vec{u}_L, \vec{u}_R) = \hat{C}(\vec{u}_L, \vec{u}_R) + \hat{I}(1,2)$.

With the "ergodic assumption", that N independent measurements on one system are equivalent to one measurement on N independent systems, we can prove

Theorem 5. *If the system is in the ontic condition*
$V_n(1,2) = \nu(\rho_n; \delta_n),\ \delta_n \ll \epsilon$, *then,*

$$|c(\vec{u}_L, \vec{u}_R)_Q(V_n) \ - \ Tr(\rho_n \hat{C}(\vec{u}_L, \vec{u}_R))| \ < \delta_n. \tag{4}$$

Proof. Let $\hat{E}(\vec{u}_L, \vec{u}_R)$ be the projection operator of $\hat{C}(\vec{u}_L, \vec{u}_R)$. Then the proof of Theorem 5 in [10] can be used with $\hat{E}(\vec{u}_L, \vec{u}_R)$ replacing the projection operator $\hat{P}_i(1)$. □

From the theory of independent errors the most probable value of a set of measurements is their arithmetical mean but $\forall n,\ |Tr\rho_n\hat{C}(\vec{u}_L,\vec{u}_R) - Tr\rho_0\hat{C}(\vec{u}_L,\vec{u}_R)| < \epsilon$ therefore the most probable value is approximately $Tr\rho_0\hat{C}(\vec{u}_L,\vec{u}_R)$.

Corollary 2. *When* $\rho_0 = \hat{P}_{\Psi(1,2;s_1,s_2)^\pm}$ *then* $Tr\rho_0\hat{C}(\vec{u}_L,\vec{u}_R) = -\vec{u}_L\cdot\vec{u}_R$.

$$\Psi(1,2;s_1,s_2)^\pm = \frac{1}{2}(\phi_L(1)\otimes\phi_R(2) \pm \phi_R(1)\otimes \phi_L(2))\otimes(|+_{s_1}\rangle\otimes|-_{s_2}\rangle + |-_{s_1}\rangle\otimes|+_{s_2}\rangle) \tag{5}$$

where L and R label opposite directions along the z-axis and $|\pm_{s_j}\rangle$ *represents particle j's spin up (down) polarisation state along a direction* $\vec{b}$ *orthogonal to the z-axis. Therefore in this qr-number deterministic model, Bell's theorem is violated in accordance with experiment.*

qr-number equations of motion for massive particles.

For any open set U, the qr-number values $(\vec{q}(U)(t),\vec{p}(U)(t))$ of the position and momentum of a massive quantum particle satisfy classical equations of motion. Thus, if $h(\vec{q}(U)(t),\vec{p}(U)(t))$ is the qr-number value of the Hamiltonian,

$$\frac{dq^j(U)(t)}{dt} = \frac{\partial h(\vec{q}(U)(t),\vec{p}(U)(t))}{\partial p^j_Q(U)(t)}, \tag{6}$$

$$\frac{dp^j_Q(U)(t)}{dt} = -\frac{\partial h(\vec{q}_Q(U)(t),\vec{p}_Q(U)(t))}{\partial q^j_Q(U)(t)}. \tag{7}$$

where $h(\vec{q}_Q(U)(t),\vec{p}_Q(U)(t)) = \sum_{j=1}^3 \frac{1}{2m}(p^j_Q(U)(t))^2 + V(\vec{q}_Q(U)(t))$.

The Hamiltonian equations of motion for $\vec{q}_Q(W)(t)$ are compared with equations for $\vec{q}(t)_Q(W)$ obtained by averaging Heisenberg operator equations over W. They give the same trajectory, $\vec{q}_Q(W)(t) = \vec{q}(t)_Q(W)$, when $V(\vec{q}_Q(U)(t))$ is linear: e.g. free and SHM motion. For suitably smooth forces, the evolutions are locally indistinguishable - there exists a class of open subsets $W(\vec{x},\epsilon) \subset \mathcal{E}_\mathcal{S}(\mathcal{A})$ on which the averaged values of Heisenberg's operator equations closely approximate the qr-number Hamilton's equations.

The class $\{W(\vec{x},\epsilon);\ \vec{x}\in\mathbb{R}^3,\ \epsilon>0\}$ do not cover $\mathcal{E}_\mathcal{S}(\mathcal{A})$, but for each $W(\vec{x},\epsilon)$ there is an open ball $B(\vec{x},\delta)\subset\mathbb{R}^3$, the collection of which cover $\mathbb{R}^3$. An observer measuring a particle with apparatus set up in one of these open balls could not determine locally whether the evolution of the particle was governed by Heisenberg's operator equations of motion averaged over $W(\vec{x},\epsilon)$ or by Hamilton's qr-number equations of motion restricted to $W(\vec{x},\epsilon)$

Summing up

The above provides a prima facie case for the hypothesis that the shift from classical to quantum theory entails a change in the type of real numbers that physical variables can have as quantitative values.

The real numbers for a quantum system are Dedekind real numbers in a spatial topos built upon the system's state space whose open sets are the domains of definition of the numbers and are the completed states (conditions) of the system. The logic is thereby intuitionistic. These changes maintain the experimentally verified predictions of standard quantum theory.

Microscopic entities possess qualities with definite qr-number values even in the absence of a specific macroscopic experimental arrangement. The qr-number values exist to extents which may be limited by the experimental arrangement or by interactions between the entities.

If the Dedekind real numbers hypothesis is correct then there should be a postulate of covariance under change of topos in which the sheaf of Dedekind real number exists because maps between $\mathbb{R}_\mathrm{D}(X)$ and $\mathbb{R}_\mathrm{D}(Y)$ are obtained using functors between the toposes $Shv(X)$ and $Shv(Y)$.

This postulate would be along the lines that the general laws of physics should be expressible in equations which hold for all systems of Dedekind real numbers. For example, the equations of motion for both quantum and classical particles can be expressed in Dedekind real numbers in both the Lagrangian and Hamiltonian formulations.

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A qr-number description of the two-slit experiment

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Abstract: *The two-slit experiment for a massive scalar particle is described using quantum real numbers (qr-numbers) as the numerical values of the particle's position and momentum. The model assigns physical reality to single quantum particles enabling it to describe the build up of the interference pattern. The qr-numbers are Dedekind real numbers in the topos of sheaves on the state space of the particle. Propositional truth values are given by open subsets of state space, each interpreted as the complete state of a single particle and called its condition. Each condition determines quantum real number values for all the particle's attributes. The spatial locations of quantum particles form a non-classical spatial continuum such that a single particle can have a trajectory that passes through two classically separated slits while not being detectable in either slit. Questions about the behaviour of single particles, unanswered in the standard quantum description, are answered.*

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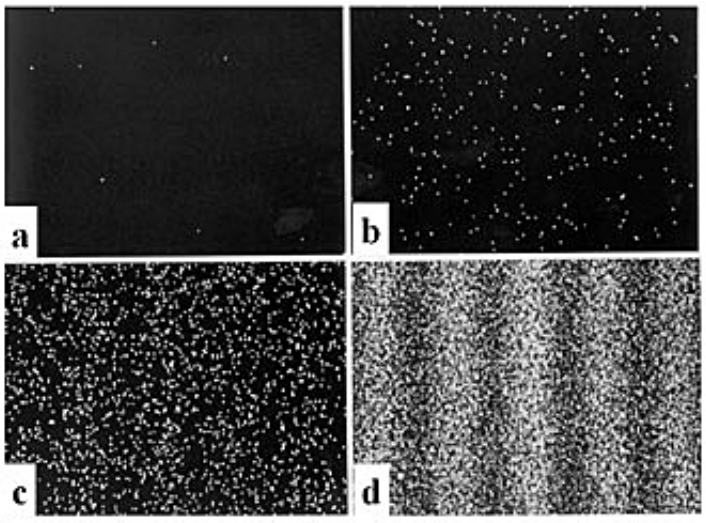
Introduction

The two-slit experiment has been one of the most important experiments for the conceptual development of quantum mechanics. Richard Feynman [16] called it "a phenomenon which is impossible ... to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the only mystery." It is here that the idea of "wave mechanics" seems most apposite. On the other hand the Schrödinger equation is not a classical hyperbolic wave equation and as Bell pointed out, just what is waving is not clear [18]. Moreover, in place of extended wave, only a sequence of discrete, localised events are detected. In the results of the Bologna (1974) [11] and Hitachi (1989) [15] double slit experiments the detection of a sequence of single electrons are detected from which an interference pattern is extrapolated. This is apparent in the Hitachi pictures.



Single-electron Build-up of Interference Pattern

Single-electron events build up over a 20 minute exposure to form an interference pattern in this double-slit experiment by Akira Tonomura and co-workers. (a) 8 electrons; (b) 270 electrons; (c) 2000 electrons; (d) 60,000 electrons.

The mystery behind this phenomenon is resolved when the quantitative values of the attributes of the electrons are taken to be not standard real numbers but Dedekind real numbers in a topos of sheaves [10] on the electron's quantum state space. Many of the conceptual mysteries of quantum mechanics come from trying to fit microscopic phenomena into a conceptual framework in which standard real numbers are assumed to be the quantitative values for the attributes of a quantum system. The argument for this is that the outcomes of experiments always are rational numbers, but the standard rationals are dense in the electron's Dedekind real numbers and hence can be obtained as the measured values of its attributes to the precision required in the experiment.

The quantum real number hypothesis:

The paradigm shift from classical to quantum theory can be seen as the change in the type of real numbers that the physical attributes of a microscopic system take as their quantitative values. This not a change in coordinates, nor a change of frame: it is a change in the underlying field of real numbers. The first step was to replace the standard real numbers by matrices, see Heisenberg [20], but this didn't distinguish between an attribute and the numerical values it takes. The real numbers for a quantum system are Dedekind real numbers in a spatial topos built upon the system's state space whose open sets are the domains of definition of the numbers they don't have all the properties of the standard reals but they have sufficient to develop differential calculus.

In this model standard quantum states exist but their central role in standard quantum mechanics is taken by open subsets of state space, that are called conditions. Conditions are "complete states" in the sense that they both generate the values of all attributes, as the real numbers are continuous functions whose domains are open sets, and they determine the solutions of the equations of motion. For any quantum system $\mathcal{S}$ the conditions are the open subsets of its quantum state space, $\mathcal{E}_{\mathcal{S}}(\mathcal{H}_S)$. Conditions are the truth values of propositions in the intrinsic logic of the sheaf $\mathrm{Shv}(\mathcal{E}_{\mathcal{S}}(\mathcal{H}_S))$, the sub-object classifier [10] $\Omega = \mathcal{O}(\mathcal{E}_{\mathcal{S}}(\mathcal{H}_S))$, where $\mathcal{O}(\mathcal{E}_{\mathcal{S}}(\mathcal{H}_S))$ is the set of open subsets of $\mathcal{E}_{\mathcal{S}}(\mathcal{H}_S)$.

A quantum system $\mathcal{S}$ with:

(i) a Hilbert space $\mathcal{H}_S$, carrying a unitary representation of a Lie group,

(ii) physical attributes represented by self-adjoint operators in an O^*-algebra $\mathcal{M}_S$ and

(iii) a state space $\mathcal{E}_{\mathcal{S}}(\mathcal{M}_S)$ of strongly positive, normalized linear functionals on $\mathcal{M}_S$,

has qr-numbers that are sections of the sheaf of germs of continuous real-valued functions in the topos $\mathrm{Shv}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}_S))$. They contain the standard real numbers as globally constant functions. In the condition W, the attribute represented by the operator $\hat{A}$ has the locally linear qr-number value $a(W) \in \mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}_S))$, where $a(\rho) = Tr\rho\hat{A};\ \forall\rho \in W$.

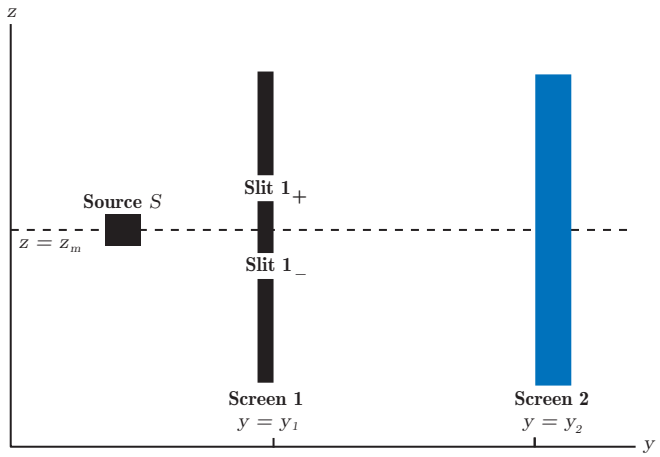
In general, for any non-empty open subset U of $\mathcal{E}_{\mathcal{S}}(\mathcal{M})$, any continuous function $F : \mathbb{R} \to \mathbb{R}$ and any locally linear function $a(U)$ defined on U, $F(a(U))$ is a qr-number defined to extent U.

Measurement

The connection between the qr-number value of an attribute and the standard rational number obtained in an experimental measurement depends on the fact that the latter is always obtained at a finite level of precision. In fact, every experiment has a confidence level, $0 \leq (1-\epsilon) \leq 1$ as well as a measure of precision $\kappa > 0$ [19], when the confidence level is unity, when $\epsilon = 0$, the experimenter is has complete confidence in the experimental results, including the measure of precision..

The correspondence between the qr-number value of a physical quantity and a standard rational number obtained in the measurement is elucidated through the process of ϵ sharp collimation [3] in a single slit. For a slit with midpoint α and width 2κ, it is achieved when the system's condition W is such that both $|a(W) - \alpha| < \kappa$ and $|a^2(W) - a(W)^2| < \epsilon\kappa^2$. This gives a necessary (but not sufficient) criterion for its qr-number value to be approximated by α with precision κ at the confidence level $1-\epsilon$.

The Heisenberg inequality implies that when the position and conjugate momentum of a particle are simultaneously measured with precision κ_z and κ_p respectively and both with confidence $(1-\epsilon)$, then the product $\kappa_z\kappa_p \geq \ \hbar/2\epsilon$. [3].



We will assume that the level of confidence for the whole double slit experiment has $0 < \epsilon < \frac{1}{4}$, a restriction that is needed for various theorems, but for most experiments, $(1-\epsilon)$ is very close to 1, so no generality is lost.

The double slit slit experiment

The set-up of the experiment

We use cartesian coordinates $\vec{x} = (x, y, z)$; the motion of the electrons is restricted to be in the y, z plane with the screens Σ_1 and Σ_2, each of thickness $2\delta y$, lying along the lines $y = y_j, j = 1, 2$ parallel to the z-axis.

The exit to S is an interval I_S lying along the z-axis given by $I_S =]z_m - \delta x_3, z_m + \delta x_3[$ for small δx_3, and the momenta of the emitted particles are limited to the intervals $J_2 =]p_y - \delta p_2, p_y + \delta p_2[$ in the y-direction and $J_3 =]-\delta p_3, \delta p_3[$ in the z-direction. The Heisenberg inequality for qr-numbers has both $\delta p_3 \delta x_3 > \frac{\hbar}{2\epsilon}$ and $\delta p_2 \delta y > \frac{\hbar}{2\epsilon}$ when both the position and momentum of an electron at S are assumed to be measurable with confidence $(1-\epsilon)$.

On Σ_1 the barrier I_0 has width $2\delta_2$, both slits $I_\pm$ have width $2\delta_1$. In the experiments $\delta_2 = 10^{-3}$mm so that the slits are much closer than they appear to be in the diagram, $p_y = 1.2 \times 10^5$km/sec and $t_2 - t_1 = 10^{-8}$sec. The electrons exiting from S whose momentum in the z-direction is less than δp_3 pass through a slit. z_m is the z-coordinate of the centre of the exit to the source S and the midpoint of the barrier I_0, $z_\pm = z_m \pm (\delta_1 + \delta_2)$ are those of the midpoints of $I_\pm$.

Questions

The standard dilemma can be summarised as follows: for an electron emitted from S that is observed to arrive on the screen Σ_2 there are two alternatives, either the electron passed through I_+ or it passed through I_-, so how can an interference pattern be built up after sufficiently many electrons have been observed on Σ_2?

This can be subdivided into three questions.

A. Why can particles that have been prepared in a like manner be detected at different spots on Σ_2?

B. How does the interference pattern emerge as the accumulated effect of many single particle events?

C. How can a single particle pass through both slits and subsequently interfere with itself?

Answers

The first point is that in the standard description all measurements are assumed to be infinitely precise: the positions of the electrons are assumed to be given by standard real numbers and their measurement yields a precise real number. In the qr-number description the positions of an electron are given by precise qr-numbers but their measurement yields a standard real number that has only a finite level of precision.

A.Because "like manner" is with reference to the parameters of the experiment. These parameters empirically determine a condition $W_S(\epsilon)$ for the ensemble of prepared electrons; individual electrons can have different conditions $V_\alpha \subset W_S(\epsilon)$, all of which satisfy the experimental restrictions but because they differ from each other follow different trajectories. We will call $W_S(\epsilon)$ the epistemic condition of the ensemble and the open sets $V_\alpha \subset W_S(\epsilon)$ are called the ontic conditions of particular electrons.

B. Because the probability function for the ensemble of particles is obtained by collating all the outcomes for all the individual particles. Since the epistemic condition is determined for the ensemble of electrons in which each electron has an ontic condition $V_\alpha \subset W_S(\epsilon)$, we can adopt an ignorance interpretation for the statistics because attributes having different qr-number values on different V_α have different trajectories in the qr-number space that produce different outcomes on being measured.

C.Because locality in qr-number space is different from locality in classical space. qr-number location has two levels:

(i) If $a|_V < z(V) < b|_V$, then the z-coordinate of the electron is weakly located in an interval $J =]a, b[$ when the electron has the condition V.

(ii)The z-coordinate of the electron is ϵ strongly located in the interval J when the electron has the condition V if

$$(1-\epsilon)|_V < \pi_z(J)(V) \leq 1|_V \tag{1}$$

$\pi_z(J)(V)$ is the qr-number value at V of the spectral projection operator, $\hat{E}^{\hat{Z}}(J)$, of the operator $\hat{Z}$ on the interval J.

If an electron in the condition V is not ϵ strongly located in the interval J then it cannot be ϵ sharp collimated in J and hence cannot be detected in J with confidence level $(1-\epsilon)$ and measure of precision $|J|/2$. This means that an electron can be weakly located in an interval without being capable of being observed.

Furthermore an electron's z-coordinate can have the qr-number value $z(V)$ where $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ and the ranges, $\mathrm{ran}z(V_1) \subset I_+$ and $\mathrm{ran}z(V_2) \subset I_-$, lie in disjoint regions $I_\pm$ of classical space. Therefore an electron could be weakly located in both slits $I_\pm$ but not ϵ strongly located in either of them and hence unable to be detected in either slit. We say that an electron that is ϵ strongly collimated in a slit I and has momentum to carry it through the slit will ϵ strongly passed through I.

Consequently for the qr-number description there are three alternatives for an electron emitted from S that is observed to arrive on the screen Σ_2: it could have ϵ strongly passed through I_+, or ϵ strongly passed through I_-, or ϵ strongly passed through $I_+ \cup I_-$ but not separately through either I_+ or I_-.

Through the slits

For an electron to pass through the slits and be detectable at the confidence level $(1-\epsilon)$, it must have a condition V an open subset of the epistemic condition $W_T(\epsilon)$ such that

$$\pi_z(I_+)(V) + \pi_z(I_-)(V) > (1-\epsilon)|_V \tag{2}$$

Therefore there are three possibilities: (a) $\pi_z(I_-)(V) < (1-\epsilon)|_V$, (b) $\pi_z(I_+(V) < (1-\epsilon)|_V$, (c) both $\pi_z(I_+)(V) < (1-\epsilon)|_V$ and $\pi_z(I_-)(V) < (1-\epsilon)|_V$. This means that in an experiment with a confidence level $(1-\epsilon)$, the epistemic condition, $W_T(\epsilon)$, of the ensemble of detectable electrons that pass through the slits is composed of three disjoint disjoint regions $W_T(\epsilon) = \bigcup_{r=\pm,m} W_r(\epsilon)$.

If $V_\alpha \subset W_T(\epsilon)$ then $x_3(V_\alpha)$, $x_2(V_\alpha)$ and $p_2(V_\alpha)$ are ϵ strongly located and yield measured values with confidence $(1-\epsilon)$. $p_3(V_\alpha)$ is only weakly located.

Theorem 1. *If $\epsilon < \frac{1}{4}$, and a particle has ontic condition V_α, then ;*
if $V_\alpha \subseteq W_+(\epsilon)$ then it can be observed to be ϵ strongly located in I_+ but not in I_-,
if $V_\alpha \subseteq W_-(\epsilon)$ then it can be observed to be ϵ strongly located in I_- but not in I_+ and
if $V_\alpha \subseteq W_m(\epsilon)$ then it can be observed to be ϵ strongly located in $I_+ \cup I_-$ but not in either I_+ or I_- separately.

It is possible that an electron could have a condition $V_\alpha = \bigcup_{r=m,\pm} V_\alpha^r$ with $V_\alpha^s = V_\alpha \cap W_s(\epsilon)$.

The method of observing the ϵ strong location in either I_+ or I_- uses a detector behind the slit, but an interference pattern is the evidence for ϵ strong location in $I_+ \cup I_-$ but not separately in either I_+ or I_-.

Trajectories in qr-number space

An electron with condition V_α leaves Σ_1 at t_1 and moves freely to arrive at Σ_2 at t_2. It arrives at the location,

$$\vec{x}(V_\alpha)(t_2) = \vec{x}(V_\alpha)(t_1) + \vec{p}(V_\alpha)(t_1)\frac{(t_2 - t_1)}{m}, \quad \vec{p}(V_\alpha)(t_2) = \vec{p}(V_\alpha)(t_1) \tag{3}$$

For the ensemble of electrons observed at Σ_2, $x_2(W_T(\epsilon))(t_k) \approx y_k$, $k = 1, 2$, $p_2(W_T(\epsilon)) \approx p_y$ and $t_2 - t_1 \approx m\frac{(y_2 - y_1)}{p_y}$ when the separation between the screens is much greater than the spread of the electrons on Σ_2.

The trajectories through $I_+ \cup I_-$.

If $V_\alpha \subset W_m(\epsilon)$, $x_3(V_\alpha)(t_1)$ is not ϵ strongly located in either of I_+ or I_- but is ϵ strongly located in $I_+ \cup I_-$. However since the distance between the slits is much less than the width of the slits, $\delta_2 \ll \delta_1$, there are two disjoint open subsets $V_\alpha^\pm \subset V_\alpha$ such that $x_3(V_\alpha^+)(t_1)$ is weakly located in I_+ and $x_3(V_\alpha^-)(t_1)$ is weakly located in I_-.

The trajectory of an electron through $I_+ \cup I_-$ has two parts, $s_\pm(V_\alpha)$,

- $(s(V_\alpha^+))$: goes from $(x_2(V_\alpha^+), x_3(V_\alpha^+))$ to $(x_2(t_2)(V_\alpha^+), x_3(t_2)(V_\alpha^+))$ with momentum $(p_2(V_\alpha^+), p_3(V_\alpha^+))$,
- $(s(V_\alpha^-))$: goes from $(x_2(V_\alpha^-), x_3(V_\alpha^-))$ to $(x_2(t_2)(V_\alpha^-), x_3(t_2)(V_\alpha^-))$ with momentum $(p_2(V_\alpha^-), p_3(V_\alpha^-))$.

The paths are observed to arrive together in the neighbourhood I_d of z_d on Σ_2, with $|I_d| = 2\delta_d$, when, if $U_d \subset W_m(\epsilon)$ is the largest open subset such that $|x_3(t_2)(U_d) - z_d 1(U_d)| < \delta_d$, while $x_2(t_2)(U_d) \approx y_2$, both $V_\alpha^\pm$ are contained in U_d. To within the experimental precision an observer would register $x_3(t_2)(V_\alpha^+)) = x_3(t_2)(V_\alpha^-))$ and deduce that

$$\begin{aligned} x_3(V_\alpha^+) - x_3(V_\alpha^-) = (p_3(V_\alpha^-) - p_3(V_\alpha^+))\frac{(t_2 - t_1)}{m} \approx \\ (p_3(V_\alpha^-)) - p_3(V_\alpha^+))\frac{(y_2 - y_1)}{p_y}. \end{aligned} \tag{4}$$

This means that the paths can't meet on Σ_2 unless the ratio of the separation between their initial z coordinates and the difference in their velocities in the z direction is equal to the time taken for the electron to go from Σ_1 to Σ_2.

If equation (4) is satisfied for V_α, then given any three of the initial qr-numbers $(x_3(V_\alpha^-), x_3(V_\alpha^+), p_3(V_\alpha^-), p_3(V_\alpha^+))$ the fourth is determined.

As $x_3(V_\alpha^\pm)$ are weakly in $I_\pm$, $x_2(V_\alpha^\pm) \approx y_1$ and $p_y \gg p_3(V_\alpha^\pm)$, the difference between the lengths of the paths, $\Delta L(V_\alpha) = L(s_+(V_\alpha)) - L(s_-(V_\alpha))$ is

$$\Delta L(V_\alpha) \approx \frac{(x_3(V_\alpha^+) - x_3(V_\alpha^-))(p_3(V_\alpha^+) + p_3(V_\alpha^-))}{2p_y}. \tag{5}$$

when they meet on Σ_2.

As the distance between the screens is much larger than the distance between the the slits, $(y_3 - y_2) \gg 2\delta_2$, if $\theta(V_\alpha^\pm)$ are the angles between the y axis and the straight line paths $s(V_\alpha^\pm)$ then $\frac{\Delta L(V_\alpha)}{(x_3^+(V_\alpha) - x_3^-(V_\alpha))} \approx |\theta(V_\alpha^+) - \theta(V_\alpha^-)|$.

Because the spread of the particles on Σ_2 is small compared with the separation between the screens the shift of the z coordinate of the meeting point from the centre point of Σ_2 is approximately

$$(y_2 - y_1)|\theta(V_\alpha^+) - \theta(V_\alpha^-)| = \frac{(p_3(V_\alpha^+) + p_3(V_\alpha^-))(y_2 - y_1)}{2p_y}. \tag{6}$$

In particular, when $(p_3(V_\alpha^+) + p_3(V_\alpha^-)) = 0$ the two parts of the trajectory meet at the midpoint of Σ_2.

Wave aspects of the ensemble of electrons

The interference pattern can only be discerned when sufficiently many electrons have been observed on Σ_2. Therefore it through physical quantities that pertain to the ensemble labelled by open subsets of $W_m(\epsilon)$ that the quantitative properties of the interference arise. For example the standard real number p_y is the measured momentum in the x_2 direction of all the electrons in the ensemble that are observed on Σ_2. This means that $p_2(W_m(\epsilon))$ must be ϵ sharply collimated in an in an interval centred on $p_y > 0$ and must be positive for the electrons to travel from Σ_1 to Σ_2.

The de Broglie relation, $\lambda_y = \frac{h}{p_y}$, between the momentum, p_y, and a length, λ_y, exists for all the electrons in this ensemble. Then the maxima occur in the pattern where $\Delta L(V_\alpha)$ is an integer multiple of λ_y, when

$$\frac{1}{2}(x_3(V_\alpha^+) - x_3(V_\alpha^-))|(p_3(V_\alpha^+) + p_3(V_\alpha^-))| = nh. \tag{7}$$

These occur on Σ_2 at locations whose z coordinates are approximately

$$\frac{(x_3(V_\alpha^+) + x_3(V_\alpha^-))}{2} \pm \frac{nh(y_2 - y_1)}{(x_3(V_\alpha^+) - x_3(V_\alpha^-))p_y} \approx z_m \pm \frac{n\lambda_y(y_2 - y_1)}{(z_+ - z_-)}. \tag{8}$$

Experimentally, the spacing between maxima
is approximately $\left(\frac{\lambda_y(y_2 - y_1)}{(z_+ - z_-)}\right)$.

Detection of particles on Σ_2

The d^{th} detector: centred at z_d, aperture I_d with $|I_d| = 2\delta_d$, registers particles whose x_3-coordinate is weakly located in I_d, but whose x_2-coordinate is ϵ strongly located in an interval $I_y =]y_2 - \delta y, y_2 + \delta y[$ while its p_2-component of momentum is ϵ sharp collimated in an interval $I_p =]p_y - \delta p, p_y + \delta p[$. Therefore by the qr-number version of Heisenberg's inequality $\delta y \geq \frac{2\hbar}{\epsilon \delta p}$ which implies that if $\epsilon < \frac{1}{\pi}$ and if $\delta p > p_y$, to ensure that electrons move from Σ_1 to Σ_2, the de Broglie wavelength λ_y equals the minimum possible half-width of the interval I_y.

If U_d is the largest open set such that $x_3(t_2)(U_d)$ is weakly located in I_d then $x_3(t_1)(U_d)$ is weakly located in an interval I_c on Σ_1 that is centred on z_d with half width $\delta_d + \frac{(t_2 - t_1)}{m}\delta p_3$. If both $I_\pm \subset I_c$ then both $V_\alpha^\pm \subseteq U_d$ and both parts of the trajectory can end in I_d if their momenta $p_3(V_\alpha^\pm)$ satisfy equation (4).

It is only when the two parts of trajectories that passed through the two slits meet in I_d on Σ_2 that an electron can be weakly located in I_d and observed, of course if an electron only passed through one of the slits and arrives in I_d it can be observed there.

The interference formula

The qr-number probability that the α^{th} electron, with condition V_α, is detected by the d^{th} detector is $\pi_z(I_d)(t_2)(V_\alpha)$.

The probability that an electron from the ensemble of electrons prepared in the empirically determined condition $W_m(\epsilon)$ is observed at the d^{th} detector is $\pi_z(I_d)(t_2)(W_m(\epsilon))$.

If the union of all conditions V_α covers $W_m(\epsilon)$, then on collating the $\pi_d(t_2)(V_\alpha)$ the probability that an electron is detected at the d^{th} detector is $\pi_d(t_2)(W_m(\epsilon)) = \cup_\alpha \pi_z(I_d)(t_2)(V_\alpha)$. This is assumed to be the case when a large ensemble is used.

The standard quantum formula: there is only one ψ_+ in $\hat{P}_+\mathcal{H}$ and only one ψ_- in $P_-\mathcal{H}$ so that there is only one vector $\psi_m = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-)$ in $P_m\mathcal{H}$. Then we may take $W_m(\epsilon) = \nu(\rho_m, \epsilon)$ with $\rho_m = |\psi_m\rangle\langle\psi_m|$.

Each $\rho \in W_m(\epsilon)$ satisfies $Tr|\rho - \rho_m| < \epsilon$. Therefore $|\pi_d(t_2)(W_m(\epsilon)) - Tr(\rho_m \hat{P}_d(t_2))| < \epsilon$ so that the qt-number probability of detecting an electron in I_d is well approximated by

$$Tr\rho_m\hat{P}_d = \tfrac{1}{2}[\langle\psi_+, \hat{P}_d\psi_+\rangle + \langle\psi_-, \hat{P}_d\psi_-\rangle + \langle\psi_+, \hat{P}_d\psi_-\rangle + \langle\psi_-, \hat{P}_d\psi_+\rangle]. \tag{9}$$

which is the standard interference formula for the two slit experiment.

Discussion

This is a part of a project, [1]- [8], to make quantum theory complete in the sense of Einstein [9] (1921), "A complete system of theoretical physics is made up of concepts, fundamental laws which are supposed to be valid for those concepts and conclusions to be reached by logical deduction. It is these conclusions which must correspond with our separate experiences;'.Some of the separate experiences of quantum-physics, mainly those associated with measurements, cannot be deduced from the laws of the standard theory. Here we discuss one such experience: the Bologna (1974) [11] and Hitachi (1989) [15] double slit experiments show single electrons building up an interference pattern. This process cannot be described using the concepts and laws of standard quantum mechanics. Richard Feynman [16] called it "a phenomenon which is impossible ... to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the only mystery [of quantum mechanics]."

The interference pattern is built up because more particles whose ontic conditions are subsets of $W_m(\epsilon)$ have qr-number trajectories that are detected on Σ_2 near points of "constructive interference" than are detected near points of "destructive interference" and my conjecture is that these are determined by the conserved angular momentum about the x axis, but I haven't been able to prove it.

This model of the experiment does not support the conjectures of C. Brukner and A. Zeilinger [17] concerning the influence of an observer on the reality of the experiment. They claimed that "The observer can decide whether or not to put detectors into the interfering path. That way, by deciding whether or not to determine the path through the two-slit experiment, he can decide which property can become reality. If he chooses not to put the detectors there, then the interference pattern will become reality; if he does put the detectors there, then the beam path will become reality. Yet, most importantly, the observer has no influence on the specific element of the world which becomes reality. Specifically, if he chooses to determine the path, he has no influence whatsoever which of the two paths, the left one or the right one, Nature will tell him is the one where the particle is found. Likewise, if he chooses to observe the interference pattern he has no influence whatsoever where in the observation plane he will observe a specific particle. Both outcomes are completely random."

Of course the observer, by deciding how to set up the apparatus will determine the type of outcome, but the qr-number model reveals that the reality is there independent of whether the observer decides to put a detector immediately behind one of the slits or not. If an electron's condition is a subset of $W_m(\epsilon)$ then it can't be detected at one slit, if it is a subset of $W_+(\epsilon)$ or $W_-(\epsilon)$ then it will be detectable at the corresponding slit. After the preparation of the electrons at the source S, the observer has no influence on the condition of a particle, which is a randomly chosen open subset of the epistemic condition $W(\epsilon)$. The observer does not decide which property can become reality, all he/she can do is to decide what part of reality to look for.

Appendices

Mathematical assumptions

Quantum real numbers

The quantum real numbers, $\mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}_S))$, for $\mathcal{S}$ have the following properties [14]:

1. The integers $\mathbb{Z}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}_S))$ and the rationals $\mathbb{Q}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}_S))$ are sheaves of locally constant functions $\mathcal{E}_{\mathcal{S}}(\mathcal{M}_S) \to \mathbb{Z}$ or $\mathbb{Q}$.

2. They have orders $<$ and $\leq$ compatible with those on $\mathbb{Q}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}_S))$, but the order $<$ is not total because trichotomy; $x > 0 \vee x = 0 \vee x < 0$, is not satisfied. The inequality $\leq$ is not equivalent to the disjunction of $<$ and $=$.

3. They are a residue field; closed under the commutative, associative, distributive binary operations $+$ and $\times$, have $0 \neq 1$ and if a number b is not invertible then $b = 0$. If $a = a(V)$ and $b = b(W)$ with $V, W \in \mathcal{O}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}_S))$, then: $a = b$ to an extent given by an open subset of $W \cap V$. If a and b are locally linear qr-numbers defined by operators $\hat{A}$ and $\hat{B}$ and they are equal on a non-empty open set then the defining operators are equal $\hat{A} = \hat{B}$, but the range of $a(W)$ being equal to the range of $a(V)$ does not imply that $W = V$.

4.The sum of $a = a(V)$ and $b = b(W)$ is defined pointwise: $a + b = c$ where c is the continuous function defined on $W \cup V$ which is equal to $a + b$ on $W \cap V$, to a on $W \setminus (W \cap V)$ and to b on $V \setminus (W \cap V)$. Similarly, their product $a \cdot b$ is defined on $W \cap V$ by $a \cdot b(x) = a(x)b(x); \forall x \in W \cap V$.

5. They have a distance function $|\cdot|$ defining a metric. It is a complete metric space with $\mathbb{Q}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}_S))$ dense in it. A number b is apart from 0 iff $|b| > 0$. It is an apartness field, i.e., $\forall b \in \mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}_S))$, $|b| > 0$ iff b is invertible.

6. They are not Dedekind complete, as least upper bounds need not exist and they are not Archimedean, because there are infinitesimal qt-numbers. For example, the expectation value $a(\rho_0) = Tr\rho_0\hat{A}$ for a given state and a given operator is an infinitesimal qr-number because there is no open set $V \neq \emptyset$ such that $a(\rho_0)|_V > 0 \vee a(\rho_0)|_V < 0$.

Quantum space

The spatial continuum in which quantum phenomena take place is not assumed to be classical, The quantum space for a quantum particle $\mathcal{S}$ is $\mathbb{R}_{\mathrm{D}}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}_S))^3$, a vector space over the globally constant classical real numbers. Cartesian coordinate axes parametrized by $\mathbb{A}^{a_Q}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}_S))$ for $a = x, y, z$, generated by the position operators $\hat{X}$, $\hat{Y}$ and $\hat{Z}$. [4]

The quantum "points" are triplets $\vec{x}_Q(W) = (x_Q(W), y_Q(W), z_Q(W))$ for $W \in \mathcal{O}(\mathcal{E}_{\mathcal{S}}(\mathcal{M}_S))$. They are not classical points, but are open sets. [10] The qr-number distance between $A = \vec{x}_Q(W)$ and $B = \vec{x}_Q(V)$ is $d_Q(A, B)(U) =$
$(|x_Q(W \cap U) - x_Q(V \cap U)|^2 + |y_Q(W \cap U) - y_Q(V \cap U)|^2 + |z_Q(W \cap U) - z_Q(V \cap U)|^2)^{1/2}$ for any open set U. If $W \cap V = \emptyset$ then $d(A, B)(U) > 0, \forall U \in \mathcal{O}(V \cup W)$, if $W \cap V \neq \emptyset$ then $d(A, B)(U) = 0$, $\forall U \in \mathcal{O}(V \cap W)$.

The d_Q defines a pseudo-metric which is a metric for quantum points that are apart. [14] Apartness is stronger than not equal to, as $\vec{x}_Q(W) \neq \vec{x}_Q(V)$ iff $W \neq V$, but $\vec{x}_Q(W)$ is apart from $\vec{x}_Q(V)$ iff $W \cap V = \emptyset$. In the quantum continuum not all pairs of different quantum points are apart.

Physical properties

Dynamics

The evolution of two quantum particles $\mathcal{S}$ and $\mathcal{M}$ with non-zero mass can be described: either
(1) by Hamiltonian equations motion for the qr-number values of the canonical position and momentum attributes of the systems. The qr-number Hamiltonian is a function of $\vec{x}^K(W(S, M)), \vec{p}^K(W(S, M))$, $K = S, M$, or
(2) by changes in the condition of the joint system controlled by $\hat{U}_t$, the Schrödinger evolution of its states, or
(3) by changes in the operators of the joint systems using Heisenberg's equations,
(2) and (3) are globally equivalent and give the same outcomes as (1) when canonical position and momentum attributes are available and the forces are linear. The linear forces include the free motion, the harmonic oscillator and the Coulomb (via Bohlin's theorem) and the von Neumann impulsive interactions, as well as the controlled shifts of Zurek . If the forces are not linear but are smooth functions of the variables $\vec{x}^K(W_0(S, M))$ then (2) and (3) have equations that closely approximate those of (1) on a collection of open sets that don't cover $\mathcal{E}_{\mathcal{S}}(\mathcal{A}_{S,M})$ but are sufficient for the collection of quantum points $\{\vec{x}^K(\tilde{W}_0(K))\}$; $K = S, M$, to cover the classical space $\mathbb{R}^3$ [2].

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Quantum law of rare events

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Abstract: *In classical physics the joint probability of a number of individually rare independent events is given by the Poisson distribution. In quantum mechanics, the same law rare events describes, for example, tunnelling escape of distinguishable particles in a double well potential trap. The situation changes drastically if the trap contains identical bosons. In this case, the classical law is modified by quantum interference between scenarios which lead to exclusive outcomes for non-identical particles. This allows, among other effects, counter flow of particles back into the densely populated state.*

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Introduction

In classical physics and statistics, probability for a number of individually rare events is universally given by the Poisson distribution (see, for instance, [1]). For example, consider a test tube divided in two parts by a penetrable membrane. Let $p(t)$ be the probability for an atom of ideal gas cross to cross the membrane from either side during a time t. Then place a large number of non-interacting classical atoms, $N >> 1$, in, say, the left half, and reduce the individual transition probability, $p(t) = w(t)/N$, keeping w finite. At any finite time, here will be atoms which have crossed the membrane from left to right, but practically no atoms recrossing from right to left. Since the probability to cross the membrane is extremely small, and the number of atoms in the right half is finite, the recrossing process is highly improbable. Should one prepare many such test tubes, the number of particles which have escaped to the right by a time t will be distributed according to the Poisson law, also known as the the *Law of Rare Events* (LRE). One might ask whether this is also always true in quantum mechanics? The question was studied in [2].

Let us replace the divided test tube by a double well potential in one dimension. While the barrier dividing the two wells is impenetrable, there are two similar energy levels, one on the right and one on the left (see the sketch in Fig.1). Such a structure can be created using modern laser techniques (see, for example, [3]). Now we can populate the level on the left with a large number of non- or weakly-interacting atoms, N, and lower the barrier so as to allow the atoms to escape into the right well by tunnelling. There are two main differences from our classical example. Firstly, quantum mechanics operates with probability amplitudes, rather than directly with the probabilities. Secondly, all atoms in the same internal state are identical bosons (we cannot of course put several fermions on the same level, so bosons they are). If N is large, and tunnelling is improbable, will the numbers of escaped atoms differ significantly from those predicted by the classical LRE?

Single-particle transition amplitudes and probabilities

We start by constructing transition amplitudes for a single atom which can occupy one of the two levels in an asymmetric double well potential. It is convenient to write the Hamiltonian in terms of the Pauli's matrices,

$$\hat{H} = \epsilon\sigma_z + \xi\sigma_x + \eta\sigma_y, \tag{1}$$

where the spin states $|1\rangle$ and $|2\rangle$, aligned up and down the z-axis, correspond to an atom occupying the right and the left well, respectively. The constants ϵ, ξ and η are at our disposal. For the evolution operator $\hat{U}(t) = \exp(-i\hat{H}t)$ we have

$$\hat{U}(t) = I\cos(\omega t) - i\omega^{-1}[\epsilon\sigma_z + \xi\sigma_x + \eta\sigma_y]\sin(\omega t), \tag{2}$$

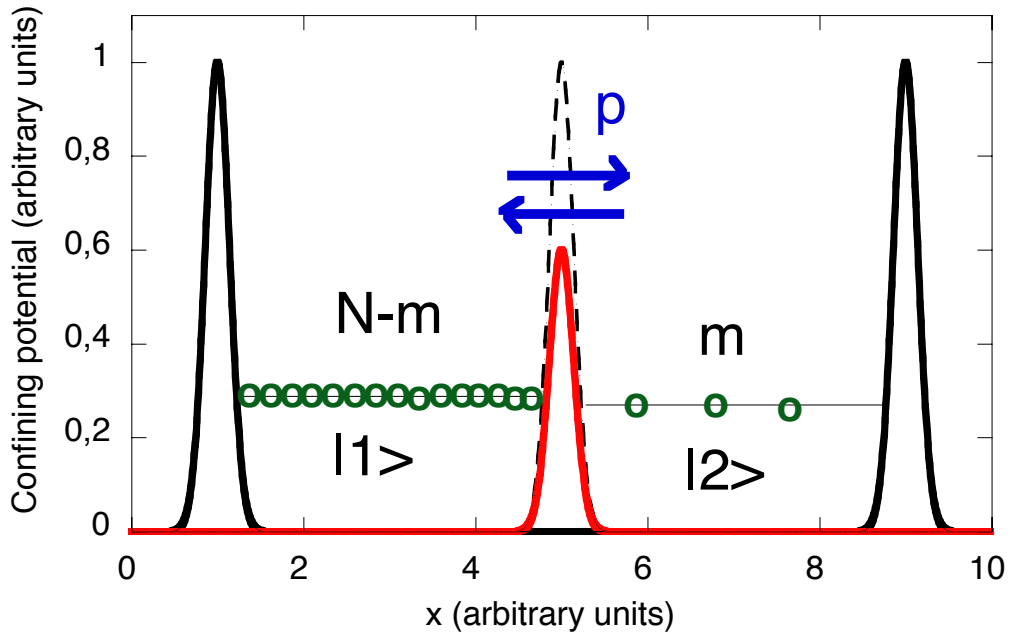

Figure 1: Double-well trap containing N atoms. The central barrier is lowered to allow tunnelling between the states $|1\rangle$ and $|2\rangle$. [2]

where we have defined $\omega = \sqrt{\epsilon^2 + \xi^2 + \eta^2}$. Its matrix elements, $U_{i,j}(t)$, are the complex valued probability amplitudes for a single atom, initially in the state $|j\rangle$, to end up in the state $|i\rangle$ after a time t. We can write them as (a star indicates complex conjugate)

$$\begin{aligned} U_{11} &= \sqrt{1-p}\exp(i\alpha) = U_{22}^* \\ U_{12} &= \sqrt{p}\exp(i\beta) = -U_{21}^* \end{aligned} \tag{3}$$

where, with our choice of the basis, $p(t) = (\xi^2 + \eta^2)\sin^2(\omega t)/\omega^2$ is the one-particle transition probability, $\alpha(t) = -\arctan[\epsilon \tan(\omega t)/\omega]$ and $\beta = -\pi/2$. Our experiment is as follows: given a total of N atoms, we put m of them in the right well, lower the barrier to allow tunnelling, and ask what is the chance of finding m' particles on the right after a time t. In other words, we wish to evaluate the transition probabilities $p^N_{m'm}(t)$ for starting with m particles in the state $|1\rangle$ and ending, after a time t, with m' particles in the same state.

The classical N-coin problem. Paths and pathways

Before considering identical bosons, it is worth looking at what would have happened had all the atoms been distinguishable. The problem is equivalent to a classical N-coin one: given that each coin changes its state with a probability p, and m coins initially heads up, what is the probability to have m' heads up after each coin has been tossed once? There is more than one way to reach the desired configuration. For $m' = m + 1$ we can flip just one coin from tails to heads. Or we can flip one coin from heads to tails, and then two tails left to heads, and so on. In general, the same result can be achieved by moving ν coins from tails to heads, and μ coins from heads to tails, provided $\nu - \mu = m' - m$. The total probability of each such process, $p^{\nu+\mu}(1-p)^{N-\mu-\nu}$ depends only on ν and μ. But then we can choose the μ particular coins to be flipped from heads to tails in C_μ^m different ways, where $C_l^k \equiv \frac{k!}{l!(k-l)!}$ is the binomial coefficient. Similarly, there are C_ν^{N-m} ways to choose the coins to be flipped the other way. Quantum mechanics often uses words lake 'paths' and 'pathways', and we will call a *path* a particular way to rearrange the coins, and a *pathway* the union of paths in which μ and ν coins change their states, as was discussed above. The probability for a pathway defined by the values (μ, ν) is then

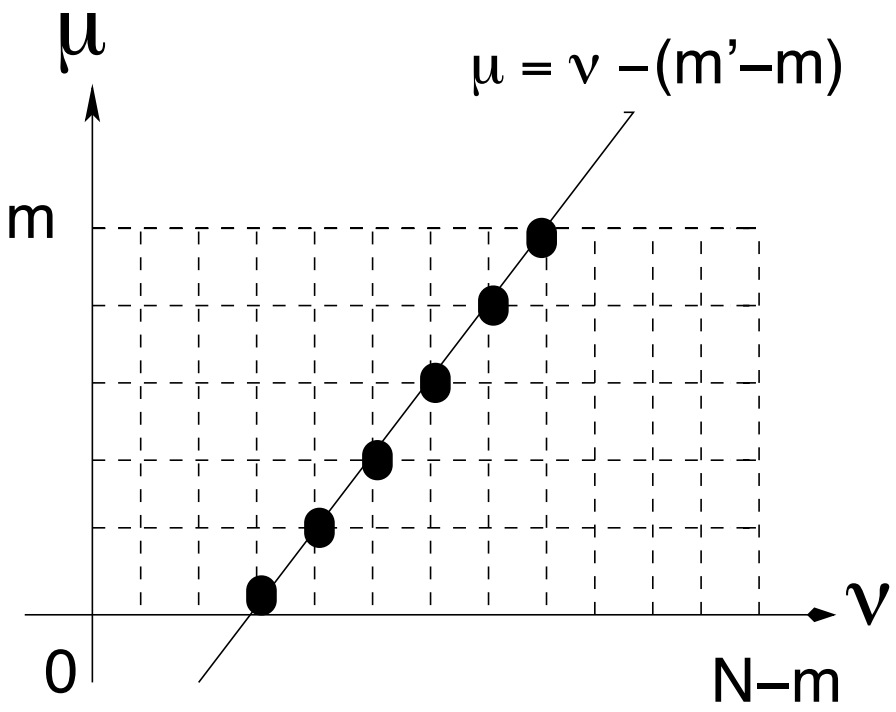

Figure 2: A diagram showing the region of summation in Eqs. (4) and (9) (filled dots). Each dot contributes $C_\mu^m C_\nu^{N-m} p^{\nu+\mu}(1-p)^{N-\mu-\nu}$ for distinguishable particles, and $C_\mu^m C_\nu^{N-m}(-1)^\mu U_{12}^{\nu+\mu} U_{11}^{m-\mu} U_{22}^{N-\mu-\mu}$ for identical bosons. [2]

$C_\mu^m C_\nu^{N-m} p^{\nu+\mu}(1-p)^{N-\mu-\nu}$. The full transition probability $p_{m'm}^N(t)$ is given by the sum over all pathways,

$$p_{m'\leftarrow m}^N = \sum_{\mu=0}^{m} \sum_{\nu=0}^{N-m} C_\mu^m C_\nu^{N-m} p^{\nu+\mu}(1-p)^{N-\mu-\nu} \delta_{\nu-\mu,m'-m} \tag{4}$$

where the Kroneker delta, δ_{mn}, ensures the correct final number of heads up. The above is illustrated in the diagram in Fig.2, where each dot corresponds to a pathway (μ, ν), and the summation in Eq.(4) is over dots lying along the line $\nu = \mu + m' - m$. Depending on N, m and m', the sum in Eq.(4) may contain a different number of terms, corresponding to various pathways connecting the initial and final states labelled by m and m', respectively.
For our distinguishable atoms any two N-particle states which differ in the position of at least one atom are orthogonal. Thus, all paths lead to distinguishable outcomes and are exclusive alternatives in the language of [4]. Summing the probabilities over the paths we see that $p_{m'm}^N(t)$ is given by Eq. (4), with the one-particle probability p defined in Eqs.(3)

The classical law of rare events

In the rare events (RE) limit we let the total number of particles in the left well, $N-m$, increase, while reducing proportionally the likelihood for each atom to tunnel

$$N \to \infty, \quad p \to w/N. \tag{5}$$

It is sufficient to retain only the leading $\mu = 0$ terms in the sum (4) which are the only non vanishing contributions. These are represented by the dots in the lowest row of diagram in Fig.2 in (4) and carry the probability weights $C_\nu^{N-m} p^\nu (1-p)^{N-\mu}$. Using the relation $lim_{N\to\infty} C_m^N = N^m/m!$ for the binomial coefficient, yields the

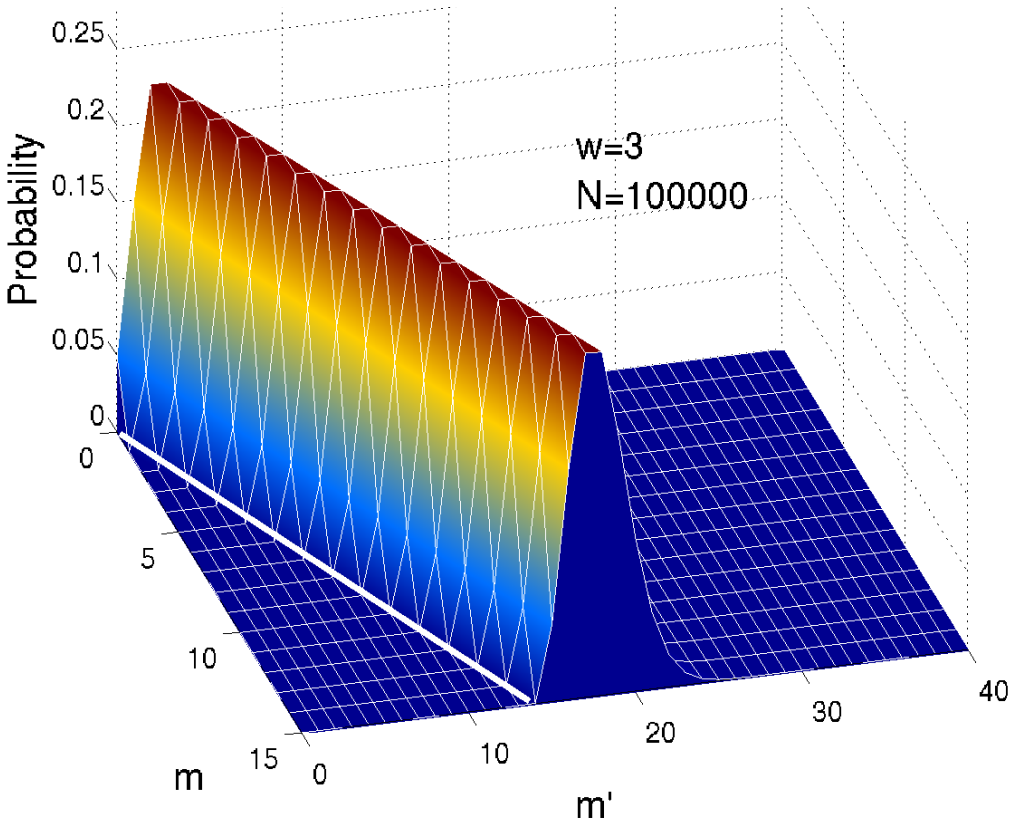

Figure 3: Poissonian probability $p^N_{m'\leftarrow m}$ for distinguishable particles, as given by Eq.(4), for $N = 10^5$ and $w = 3$. The white line marks $m = m'$. [2]

Poisson distribution,

$$\lim_{N\to\infty} p^N_{m'\leftarrow m} = \begin{cases} w^q \exp(-w)/q! & q \equiv m' - m \geq 0 \\ 0 & q < 0, \end{cases} \tag{6}$$

shown in Fig.3.
This is what one would expect. Consider for example a symmetric trap with $\epsilon = 0$. Reducing the transition probability in (5) we also make the Rabi period $2\pi/\omega$ after which the system must return to its initial state, extremely large. Now, for all $t << 2\pi/\omega$, escape of atoms into the right trap can be considered practically irreversible, with the number of particles arriving there, q, independent of the number of particles, m, already there. Low probability of each individual event, and much lower population in the right well make re-crossings from right to left statistically insignificant (see Fig.3). In particular, after detecting m particles in the right well, one never finds it empty again, as the probability $p^N_{0\leftarrow m}$ (the left upper corner in the diagram in Fig. 2), vanishes as $(w/N)^m \exp(-w)$. One might expect a similar argument to apply if the non-identical particles are replaced with non-interacting bosons. Next we will show that this is not the case.

Identical bosons and a quantum N-coin problem

For identical particles, there is no way to distinguish the states in which, say, the first or the second boson has escaped into the right well. It is not that one does not know which of the two processes has occurred. One *cannot* know, as the the two scenarios now lead to the same final state, and are interfering alternatives in the language of [4]. In quantum mechanics two interfering paths cannot be distinguished. Such paths should be considered a single route, and their amplitudes, rather than the probabilities, must be added.
Thus, for identical bosons we have a quantum version of the N-coin problem: after a toss each coin changes its

state from $|i\rangle$ to $|j\rangle$, $i,j=1,2$ with the probability amplitude U_{ji}, and we must sum amplitudes rather than probabilities over all pathways leading to the same final outcome. The state of the system with any m coins displaying heads is given by a symmetrised wave function

$$|m,N\rangle = (C_m^N)^{-1/2}\sum\prod_{j=1}^{N}|i_j\rangle_j, \quad i_j=1,2 \tag{7}$$

where $|i\rangle_j$, $i=1,2$ indicated the state if the j-th particle, and the sum is over C_m^N different ways to ascribe to m of the N indices i_j the value of 2, and to the remaining $N-m$ ones the value of 1. After all coins are tossed once each individual term in the sum of Eq.(1) contributes to the amplitude to have m' heads up a quantity

$$\begin{aligned} f(m'\leftarrow m,N) &= (C_m^N)^{-1/2}(C_{m'}^N)^{-1/2} \\ &\times\sum_{\mu=0}^{m}\sum_{\nu=0}^{N-m} C_\mu^m C_\nu^{N-m} U_{12}^\nu U_{21}^\mu U_{11}^{m-\mu} U_{22}^{N-m-\nu}\delta_{m'-m,\nu-\mu}, \end{aligned} \tag{8}$$

with the region of summation shown in Fig.2. Since Eq.(1) contains C_m^N such terms, the probability $P^N_{m'\leftarrow m}$ to have m' heads up after the toss is (we are using a capital P to distinguish it from the classical probability (4))

$$P^N_{m'\leftarrow m} = (C_m^N)^2|f(m'\leftarrow m,N)|^2, \quad m,m'=0,1,...N.$$

It turns out that $P^N_{m'\leftarrow m}$ can be expressed in terms of the Jacobi polynomials $\mathcal{P}_n^{(\alpha,\beta)}(x)$ [7]–[9]

$$\begin{aligned} P^N_{m'\leftarrow m} = \frac{m!(N-m)!}{m'!(N-m')!}&p^{m'-m}(1-p)^{N-m'+m} \\ &|\mathcal{P}_m^{(N-m'-m,m'-m)}(2p-1)|^2. \end{aligned} \tag{9}$$

As in the case of distinguishable particles, $P^N_{m'\leftarrow m}$ depends only on the one-particle transition probability p, and not of the phases α and β of the matrix elements of of U_{ij} in Eq.(3). It is shaped by the interference between the pathways shown in Fig.2. Interesting effects are possible due to the minus sign in the second of Eqs.(3), as the amplitudes for moving different number of particles from right to left may have different signs.
We note also that in the special case of tunnelling into an initially empty well, $m=0$, there is only one pathway corresponding to moving exactly m particles from left to right. In the absence of interference. transition probabilities for distinguishable particles and identical bosons coincide,

$$P^N_{m'\leftarrow 0} = p^N_{m'\leftarrow 0} = C^N_{m'}p^{m'}(1-p)^{N-m'}, \tag{10}$$

as was pointed out earlier in the Refs. [5] and [6].

The quantum law of rare events

More interesting, however, are the transitions affected by the interference effects which, as we will demonstrate, persist even in the RE limit (5). Since the sum in Eq.(2) contains $\sqrt{p}$ which decreases with N slower that p, (and the same statistical weights $C_\mu^m C_\nu^{N-m}$), we cannot limit ourselves to just the $\mu=0$ terms as $N\to\infty$ and $p\to 0$. With several terms retained in the sum (2), the interference is still possible, and as we will show next, the result is very different for the classical LRE (6)
Thus, after taking the limit (5), we have ($q=m'-m$)

$$\lim_{N\to\infty} P^N_{m'\leftarrow m} = \ w^q\exp(-w)|\sum_{\mu=max[0,-q]}^{m}\frac{\sqrt{m'!m!}(-w)^\mu}{\mu!(m-\mu)!(q+\mu)!}|^2, \tag{11}$$

shown in Fig.4. Equation (11), which is our central result[1], replaces the classical Poisson law (6) for non-interacting identical bosons. Some of its properties are counterintuitive.

[1]Equation (11) can be simplified further by exploring the asymptotic behaviour of Jacobi polynomials in the limit (5) e.g., by the methods of Ref. [9], but we will not pursue it further.

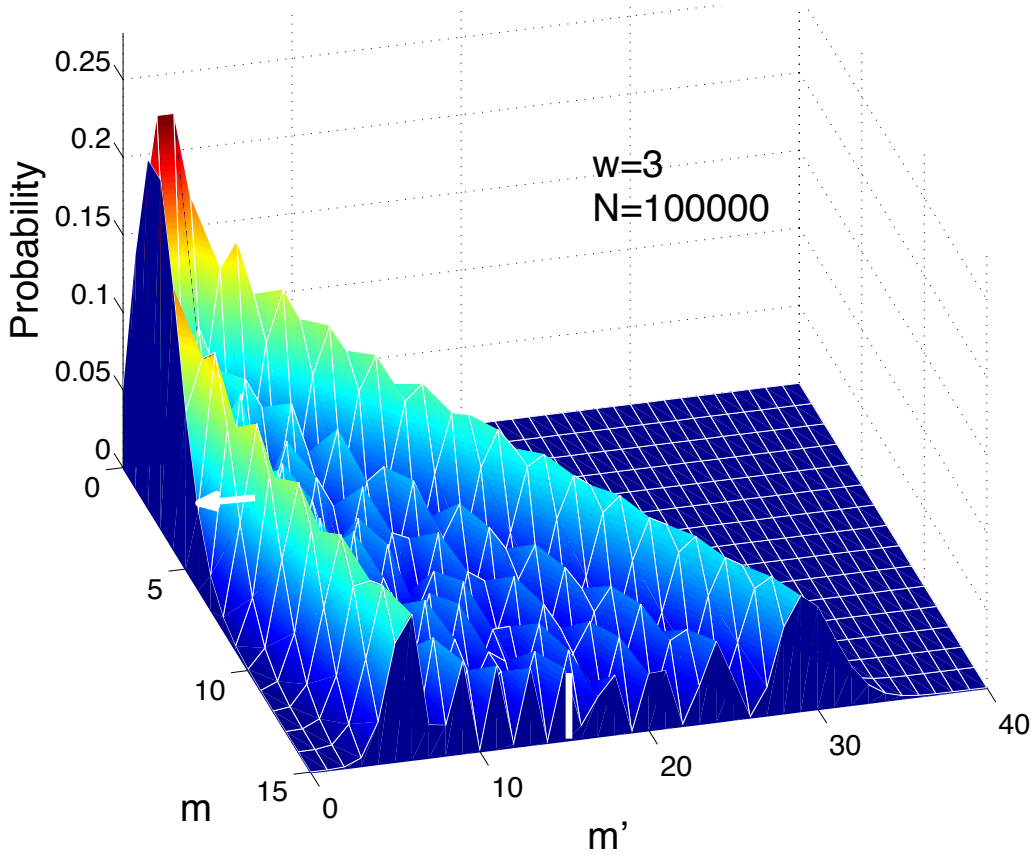

Figure 4: (Color online) Non-poissonian probability $P^N_{m'\leftarrow m}$ for identical bosons, as given by Eq.(9), for $N = 10^5$ and $w = 3$. The section of the surface indicate by arrow corresponds to $P^N_{0\leftarrow m}$ also shown in Fig.4. [2]

Unlike the Poisson distribution (6), $P^N_{m'm}$ Eq.(11) is highly structured, as a results of the interference between the pathways shown in Fig.2.
There is a much broader spread in the final number of escaped atoms for a given m.
Finally, it allows for a total or partial recapture of the few particles initially held in the right well into the left densely populated well, contrary to the simple argument based on improbability of such an event. The probability $P^N_{0\leftarrow m}$, which corresponds to the left face of the surface in Fig.4, is seen to peak at $m \approx w$. In general, the probability for all m bosons to cross into the left well, $P^N_{0\leftarrow m}$, contains only one term in the sum (11) [the left upper corner, $(\mu = m, \nu = 0)$, in the diagram in Fig. 2]. After taking the limit (5) we find that

$$P^N_{0\leftarrow m} = w^m \exp(-w)/m!, \tag{12}$$

is a Poisson distribution shown in Fig.5. (Here we stop briefly to admire the resilience of the Poisson's law which, after being dismissed from Eq.(11), immediately reappears in Eq.(12) in its new role.) The complete recapture process exhibits certain preference for the number of atoms to be readmitted into the left well, and is most likely for $m \approx w$, i.e., for m equal to the mean number of distinguishable particles which would cross into the right well under the same conditions.
To obtain the probability of a partial recapture process, such that 1, 2, ...$(m-1)$ atoms remain in the right well at the time of observation, we must cut the surface in Fig.4 along the line $m' = 1, 2, 3....$ These probabilities are structured, as shown in Fig.6. For $m > 1$ and $m' = 1$, there are two interfering scenarios leading to just one particle being left in the right well [points $(\mu = m-1, \nu = 0)$ and $(\mu = m, \nu = 1)$ in Fig.2]. The corresponding probability $P^N_{1\leftarrow m}$ is bimodal as shown in Fig. 6 by a dashed line.
The no-change probability to retain the same number of atoms in the right well, $P^N_{m\leftarrow m}$, builds up from $m+1$ interfering terms and also shows an oscillatory pattern shown in Fig.6 (long dashed).

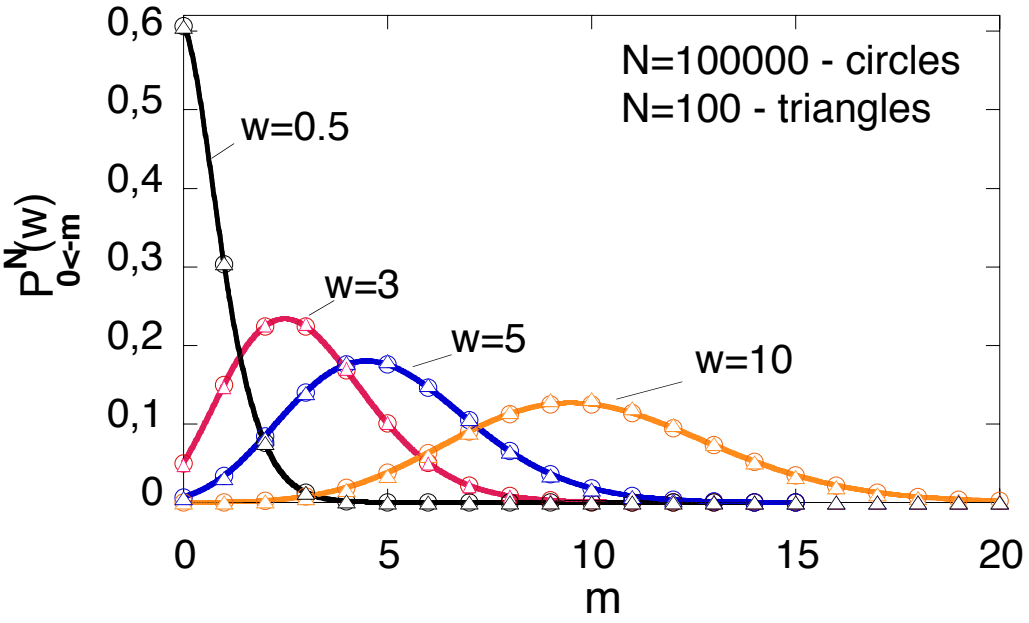

Figure 5: The probability $P^N_{0\leftarrow m}$ in Eq.(9) that all m particles initially in the right well, $m << N$, $N >> 1$, would cross to the left, leaving the right well empty. The solid lines are the corresponding Poisson distributions (12). [2]

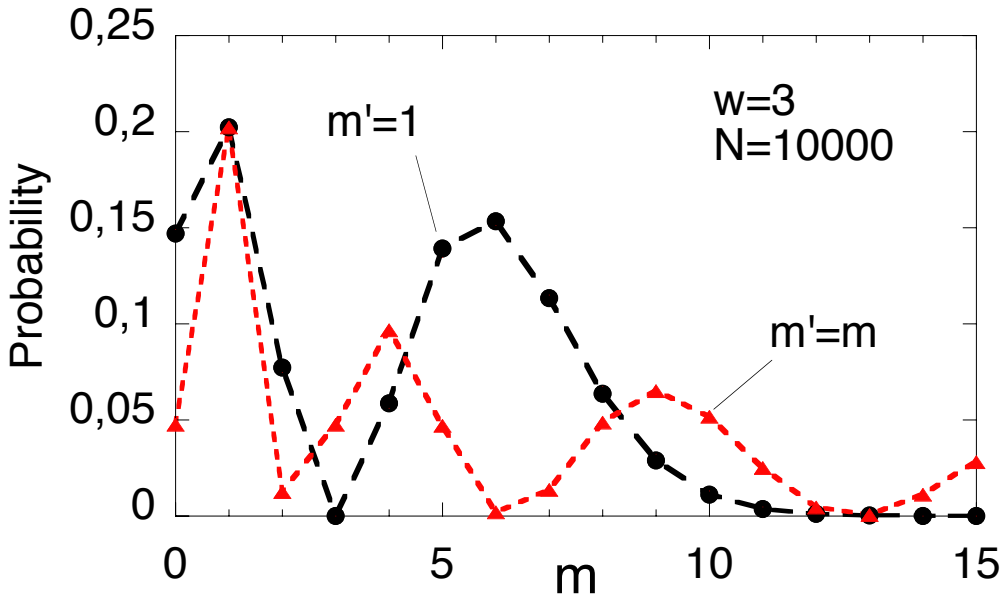

Figure 6: The probabilities $P^N_{1\leftarrow m}$ to end up with just one atom in the right well (circles, long dashed), and $P^N_{m\leftarrow m}$, to leave the population of the right well unchanged (triangles, dashed), as functions of m. [2]

Conclusion and discussion

For a brief summary, consider two states available to N particles. There are many ways (scenarios) to rearrange the particles changing their number form m to m' in, say, the first state. In quantum mechanics, all such scenarios are exclusive if the particles are distinguishable. If N is large, and each particle is unlikely to change its state, one sums the probabilities to obtain the classical Poisson law of rare events (6).
If the particles are bosons, the scenarios interfere. Summing the amplitudes, rather than probabilities, one obtains the quantum law of rare events (11), which differs significantly from its classical counterpart (6). For example, instead of always escaping from the densely populated state, the bosons may return to it.
Seen differently, even non-interacting bosons are not entirely independent and must exhibit some collective behaviour owing to the symmetry of the bosonic wave function. The symmetry is responsible, for example, for the 'bunching' effects in statistical properties of bosonic systems [10]- [14]. This collectivism makes possible the return of a few escapees to the state where most of their comrades are held. In is interesting to note, however, that this effect persist even in the limit when the escapes are infrequent and improbable, that is, in the limit of rare events.

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Stochastic Representation of Quantum Mechanics and Solitons

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Abstract: *Stochastic realization of the wave function in quantum mechanics, with the inclusion of soliton representation of extended particles, is discussed. Entangled solitons construction being introduced in the nonlinear spinor field model, the Einstein–Podolsky–Rosen (EPR) spin correlation is calculated and shown to coincide with the quantum mechanical one for the* 1/2*-spin particles.*

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Introduction. Wave–particle dualism and solitons

As a first motivation for introducing stochastic representation of the wave function let us consider the de Broglie plane wave

$$\psi = A\mathrm{e}^{-\mathrm{i}kx} = A\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}(\mathbf{kr})} \tag{1}$$

for a free particle with the energy ω, momentum $\mathbf{k}$, and mass m, when the relativistic relation

$$k^2 = \omega^2 - \mathbf{k}^2 = m^2 \tag{2}$$

holds (in natural units $\hbar = c = 1$).

Suppose, following L. de Broglie [1] and A. Einstein [2], that the structure of the particle is described by a regular bounded function $u(t, \mathbf{r})$, which is supposed to satisfy some nonlinear equation with the Klein–Gordon linear part. Let $\ell_0 = 1/m$ be the characteristic size of the soliton solution $u(t, \mathbf{r})$ moving with the velocity $\mathbf{v} = \mathbf{k}/\omega$.

Now it is worth-while to underline the remarkable fact behind this research [3], namely, the possibility to represent the de Broglie wave (1) as the sum of solitons located at nodes of a cubic lattice with the spacing $a \gg \ell_0$:

$$A\mathrm{e}^{-\mathrm{i}kx} = \sum_{\mathbf{d}} u(t, \mathbf{r} + \mathbf{d}), \tag{3}$$

where $\mathbf{d}$ marks the positions of lattice nodes. To show the validity of (3) one can take into account the asymptotic behavior of the soliton solution in its tail region:

$$u(x) = \int \mathrm{d}^4k\, \mathrm{e}^{-\mathrm{i}kx} g(k)\delta(k^2 - m^2) \tag{4}$$

and then use the well-known formula

$$\sum_{\mathbf{d}} \mathrm{e}^{\mathrm{i}(\mathbf{k}\,\mathbf{d})} = \left(\frac{2\pi}{a}\right)^3 \delta(\mathbf{k}), \tag{5}$$

implying that

$$A = \left(\frac{2\pi}{a}\right)^3 \frac{g(m)}{2m}.$$

The formula (3) gives a simple illustration of the wave-particle dualism, showing that the de Broglie wave characterizes the assemblage of particles-solitons.

D. Bohm's principle of nonlinear resonance and its gravitational mechanism

As a point of departure we consider the following problem posed by D. Bohm. Many years ago he discussed in his book [4] the possible relation between the wave-particle dualism in quantum mechanics and nonlinearity of fundamental equations in future theory of elementary particles. To represent the line of D. Bohm's thought, let us consider in Minkowsky space-time a simple scalar field model given by the Lagrangian density

$$\mathcal{L} = \partial_i \phi^* \partial_j \phi \, \eta^{ij} - (mc/\hbar)^2 \, \phi^* \phi + F\left(\phi^* \phi\right). \tag{6}$$

Here ϕ designates complex scalar field, i, $j = 0, 1, 2, 3$; $\eta^{ij} = \mathrm{diag}(1, -1, -1, -1)$, and the nonlinear function $F(s)$ behaves at $s \to 0$ as s^n, $n>1$, to guarantee the existence of particle-like solutions to the corresponding field equations, that is describing localized regular configurations possessing finite energy. In particular, the choice $F(s) = g\, s^{3/2}$, $g>0$, in (6) corresponds to the well-known Synge model [5], which is popular in nuclear physics and admits stationary radial solutions of the form

$$\phi_0 = u(r) \exp(-\mathrm{i}\omega t), \quad r = |\mathbf{r}| \quad . \tag{7}$$

The radial function $u(r)$ in (7) is regular and exponentially decreases at space infinity, thus implying the finiteness of the energy

$$E = \int \mathrm{d}^3 x \, T_0^0(\phi_0), \tag{8}$$

where T_j^i stands for the energy-momentum tensor of the field model in question. Moreover, it can be shown that the unnodal configuration, for which $u(r) >0$, turns out to be stable in the Liapunov's sense, if the charge of the configuration is fixed [6]. This fact implies the existence of slightly perturbed soliton solutions similar to (7):

$$\phi = \phi_0 + \xi(t, \mathbf{r}). \tag{9}$$

It should be stressed that the perturbation ξ in (9) appears to be small with respect to ϕ_0 in the region of soliton's localization only, though in the "tail" region of the soliton (i.e. far from its center) the function ϕ_0 is small, so one can put $\phi = \xi$.

D. Bohm posed the following question: Does there exist any nonlinear field model, for which the asymptotic behavior of the perturbed soliton solution, at large distances from the soliton's center, would represent the oscillations with the characteristic frequency $\omega = E/\hbar$? In other words, for the model in question the principal Fourier amplitude of the field $\phi \approx \xi$ at large distances $r \to \infty$ should correspond to the frequency ω related to the soliton's energy (8) via the Planck—de Broglie formula

$$E = \hbar\omega. \tag{10}$$

This property will be called **the Bohm's principle of nonlinear resonance**.

As one can see from (6), the field equation at space infinity, where $\phi \to 0$, reduces to the linear Klein—Gordon equation

$$\left(\Box - (mc/\hbar)^2\right) \phi = 0. \tag{11}$$

Therefore, the relation (10) can be satisfied for the solitons with the single energy $E = mc^2$, determined by the fixed mass m represented in (6). Thus, we conclude that the universality of the Planck—de Broglie relation (10) appears to be broken for the model (6), that forces us to modify the latter one. Taking into account that the frequency in (10) is determined by the mass of the localized system, it seems natural to use in the new modified model the proper gravitational field of the particle-soliton, in view of the fact that its asymptotic behavior at space infinity is also determined by the mass of the system. Finally, it is suggested to search for the answer to the Bohm's question in the self-consistent gravitational theory [7, 10].

The new model will be described by the Lagrangian density $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_m$, where $\mathcal{L}_g = c^4R/(16\pi G)$ corresponds to the Einstein gravitational theory and $\mathcal{L}_m$ is written as follows:

$$\mathcal{L}_m = \partial_i\phi^*\partial_j\phi\, g^{ij} - I(g_{ij})\phi^*\phi + F(\phi^*\phi). \tag{12}$$

The crucial point in this scheme is the constructing of the invariant $I(g_{ij})$, which should depend on the metric tensor g_{ij} of the Riemannian space-time in such a manner that in the vicinity of the soliton with a mass m the following relation took place:

$$\lim_{r\to\infty} I(g_{ij}) = (mc/\hbar)^2. \tag{13}$$

It can be easily seen that due to (13) one finds at space infinity the universal equation (11), which is valid for the soliton configuration with an arbitrary mass m.

To show the existence of the invariant I with the property (13), one could construct it through the Riemann curvature tensor R_{ijkl} and its covariant derivatives $R_{ijkl;n}$:

$$I = (I_1^4/I_2^3)c^6\hbar^{-2}G^{-2}, \tag{14}$$

where G stands for the Newton gravitational constant and invariants I_1, I_2 have the form:

$$I_1 = R_{ijkl}R^{ijkl}/48, \quad I_2 = -R_{ijkl;n}R^{ijkl;n}/432.$$

Calculating R_{ijkl} and invariants I_1, I_2 via the Schwarzschild metric at large distance r from the soliton's center, that seems reasonable for the isolated island-like systems, one finds

$$I_1 = G^2m^2/(c^4r^6); \quad I_2 = G^2m^2/(c^4r^8). \tag{15}$$

Thus, the relations (14) and (15) imply the desirable property (13) and the validity of the Bohm's principle of nonlinear resonance in its gravitational realization, that is the Planck–de Broglie wave-particle dualism relation (10) holds for all massive particles described by regular localized field configurations.

Now the next problem arises: to prove the consistency of the Einstein–de Broglie solitonian scheme, complemented by the Bohm's nonlinear resonance principle, with the main axioms of quantum mechanics. This problem was discussed in the works [8, 13] and it was shown that in the limit of point-like particles the main quantum postulates could be retained. In particular, it turned out that on the base of solitonian field configurations one could build the analog of the probability amplitude (wave function) and the mean values of physical observables could be calculated as scalar products in a suitable Hilbert space with the stochastic properties.

Random Hilbert space

The shortest way to get the stochastic representation of quantum mechanics is to modify the formula (3). This can be easily performed if one admits that the locations of solitons' centers are not regular nodes of the cubic lattice but some randomly chosen points. To realize this prescription, suppose that a field ϕ describes n particles-solitons and has the form

$$\phi(t,\mathbf{r}) = \sum_{k=1}^{n} \phi^{(k)}(t,\mathbf{r}), \tag{16}$$

where

$$\operatorname{supp}\phi^{(k)} \cap \operatorname{supp}\phi^{(k')} = 0, \quad k \neq k',$$

and the same for the conjugate momenta

$$\pi(t,\mathbf{r}) = \partial\mathcal{L}/\partial\phi_t = \sum_{k=1}^{n} \pi^{(k)}(t,\mathbf{r}), \quad \phi_t = \partial\phi/\partial t.$$

Let us define the auxiliary functions

$$\varphi^{(k)}(t,\mathbf{r}) = \frac{1}{\sqrt{2}}(\nu_k \phi^{(k)} + \mathrm{i}\pi^{(k)}/\nu_k) \tag{17}$$

with the constants ν_k satisfying the normalization condition

$$\hbar = \int \mathrm{d}^3x\, |\varphi^{(k)}|^2. \tag{18}$$

Now we define the analog of the wave function in the configurational space $\mathbb{R}^{3n} \ni \mathbf{x} = \{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$ as

$$\Psi_N(t,\mathbf{r}_1,\ldots,\mathbf{r}_n) = (\hbar^n N)^{-1/2} \sum_{j=1}^{N} \prod_{k=1}^{n} \varphi_j^{(k)}(t,\mathbf{r}_k), \tag{19}$$

where $N \gg 1$ stands for the number of trials (observations) and $\varphi_j^{(k)}$ is the one-particle function (7) for the j–th trial.

Now we intend to show that the quantity

$$\rho_N = \frac{1}{(\triangle\vee)^n} \int\limits_{(\triangle\vee)^n \subset \mathbb{R}^{3n}} \mathrm{d}^{3n}x\, |\Psi_N|^2,$$

where $\triangle\vee$ is the elementary volume which is supposed to be much greater than the proper volume of the particle ${\ell_0}^3 = \vee_0 \ll \triangle\vee$, plays the role of coordinate probability density. To this end let us calculate the following integral:

$$(\triangle\vee)^n \rho_N \equiv \int\limits_{(\triangle\vee)^n} \mathrm{d}^{3n}x\, |\Psi_N|^2 = (\hbar^n N)^{-1} \left(\sum_{i=1}^{N} a_{ii} + \sum_{i\neq j=1}^{N} a_{ij} \right),$$

where the denotation is used

$$a_{ij} = \tfrac{1}{2} \prod_{k=1}^{n} \int\limits_{\triangle\vee} \mathrm{d}^3x \left(\varphi_i^{*(k)} \varphi_j^{(k)} + \varphi_j^{*(k)} \varphi_i^{(k)} \right).$$

Taking into account (19), one gets

$$(\triangle\vee)^n \rho_N = (\hbar^n N)^{-1} (\hbar^n \triangle N + S), \quad S = \sum_{i\neq j} a_{ij}, \tag{20}$$

with $\triangle N$ standing for the number of trials for which the centers of particles-solitons were located in $(\triangle\vee)^n$.

It is worth-while to remark that due to independence of trials and arbitrariness of initial data and, in particular, of the phases of the functions $\varphi_i^{(k)}$, one can consider the entities a_{ij} for $i \neq j$ as independent random variables with zero mean values. This fact permits one to use the Chebyshev's inequality [14] for estimating the probability of the events, for which $|S|$ surpasses $\hbar^n \triangle N$:

$$P(|S| > \hbar^n \triangle N) \leq (\hbar^n \triangle N)^{-2} \langle S^2 \rangle. \tag{21}$$

On the other hand, in view of trials' independence one gets

$$\langle S^2 \rangle = \sum_{i\neq j} \langle a_{ij}^2 \rangle. \tag{22}$$

Now one can take into account that the wave packets $\varphi_i^{(k)}$ are effectively overlapped if their centers belong to the proper volume domain $\vee_0$. This property permits to deduce from (19) and (22) the estimate

$$\langle S^2 \rangle \leq \alpha^n \hbar^{2n} \frac{\triangle N}{(\triangle\vee)^n} {\vee_0}^n \triangle N, \tag{23}$$

where $\alpha \sim 1$ is the "packing" factor for the nearest neighbors. Inserting (23) into (22), one finds the following estimate:

$$P\left(|S| > \hbar^n \triangle N\right) < \left(\alpha \vee_0 / \triangle \vee\right)^n \ll 1. \tag{24}$$

Applying the estimate (24) to (20), one can state that with the probability close to unity the following relation holds:

$$\left(\triangle \vee\right)^n \rho_N = \triangle N / N, \tag{25}$$

signifying that the construction (19) plays the role of the probability amplitude for the coordinate distribution of solitons' centers, with ρ_N in (25) being the corresponding probability density.

Now let us consider the measuring procedure for some observable A corresponding, due to E. Noether's theorem, to the symmetry group generator $\hat{M}_A$. For example, the momentum $\mathbf{P}$ is related with the generator of space translation $\hat{M}_P = -i \bigtriangledown$, the angular momentum $\mathbf{L}$ is related with the generator of space rotation $\hat{M}_L = \mathbf{J}$ and so on. As a result one can represent the classical observable A_j for the j-th trial in the form

$$A_j = \int \mathrm{d}^3 x \, \pi_j \mathrm{i} \hat{M}_A \phi_j = \sum_{k=1}^{n} \int \mathrm{d}^3 x \, \varphi_j^{*(k)} \hat{M}_A^{(k)} \varphi_j^{(k)}.$$

The corresponding mean value is

$$\begin{aligned} \mathbb{E}(A) &\equiv \frac{1}{N} \sum_{j=1}^{N} A_j = \frac{1}{N} \sum_{j=1}^{N} \sum_{k=1}^{n} \int \mathrm{d}^3 x \, \varphi_j^{*(k)} \hat{M}_A^{(k)} \varphi_j^{(k)} \\ &= \int \mathrm{d}^{3n} x \, \Psi_N^* \hat{A} \Psi_N + O(\frac{\vee_0}{\triangle \vee}), \end{aligned} \tag{26}$$

where the Hermitian operator $\hat{A}$ reads

$$\hat{A} = \sum_{k=1}^{n} \hbar \hat{M}_A^{(k)}. \tag{27}$$

Thus, up to the terms of the order $\vee_0 / \triangle \vee \ll 1$, we obtain the standard quantum mechanical rule (26) for the calculation of mean values [8, 9].

It is interesting to underline that the solitonian scheme in question contains also the well-known spin-statistics correlation [10]. Namely, if $\varphi_j^{(k)}$ is transformed under the rotation by irreducible representation $D^{(J)}$ of $SO(3)$, with the weight J, then the transposition of two identical extended particles is equivalent to the relative 2π–rotation of $\varphi_j^{(k)}$, that gives the multiplication factor $(-1)^{2J}$ in Ψ_N. To show this property, suppose that our particles are identical, i.e. their profiles $\varphi_j^{(k)}$ may differ in phases only. Therefore, the transposition of the particles with the centers at $\mathbf{r}_1$ and $\mathbf{r}_2$ means the π–rotation of 2–particle configuration around the median axis of the central vector line $\mathbf{r}_1 - \mathbf{r}_2$. However, due to extended character of the particles, to restore the initial configuration, one should perform additional proper π–rotations of the particles. The latter operation being equivalent to the relative 2π–rotation of particles, one concludes that it results in aforementioned multiplication of Ψ_N by $(-1)^{2J}$. Under the natural supposition that the weight J is related with the spin of particles-solitons, one infers that the many–particles wave function (19) should be symmetrical under the transposition of the two identical particles if the spin is integer, but antisymmetrical if the spin is half-integer (the Pauli principle).

Thus, we conclude that in the solitonian scheme the spin—statistics correlation stems from the extended character of particles—solitons. However, the particles in quantum mechanics being considered as point-like ones, it appears inevitable to include the transpositional symmetry of the wave function as the first principle (cf. Hartree—Fock receipt for Fermions).

It can be also proved that Ψ_N up to the terms of order $\vee_0 / \triangle \vee$ satisfies the standard Schrödinger equation [10]. To this end it is worth-while to underline that, in accordance with the Bohm's nonlinear resonance principle (13), in the vicinity of the k-th particle the Klein–Gordon equation (11) with the particle's mass m_k is satisfied.

However, at large distances the same equation (11) is valid but with the mass M, equal to the total mass of the system. In view of this fact, it is useful to divide the field configuration $\varphi^{(k)}$ into two parts as follows:

$$\varphi^{(k)} = \varphi_0^{(k)} + \varphi_\infty^{(k)}, \tag{28}$$

where $\varphi_0^{(k)}$ describes the nearest structure (highly decreasing function) and $\varphi_\infty^{(k)}$ describes the far one (slightly decreasing function). According to (11), in the proper reference frames of the k–th particle and of the total system respectively, one finds the following time behavior of these functions:

$$\varphi_0^{(k)} \sim \mathrm{e}^{-\mathrm{i}m_k c^2 t/\hbar}, \quad \varphi_\infty^{(k)} \sim \mathrm{e}^{-\mathrm{i}Mc^2 t/\hbar}. \tag{29}$$

Inserting (28) in (19), one gets for $r_j \to \infty$

$$\prod_{k=1}^{n} \varphi^{(k)} = \prod_{k=1}^{n} \left(\varphi_0^{(k)} + \varphi_\infty^{(k)}\right) \approx \varphi_\infty^{(k)} \prod_{k\neq j} \varphi_0^{(k)}. \tag{30}$$

In view of (29) and (30) one concludes that at $r_j \to \infty$

$$\Psi_N \sim \mathrm{e}^{-\mathrm{i}Mc^2 t/\hbar}. \tag{31}$$

On the other hand, given the field Hamiltonian $H[\phi, \pi]$ of the system, one can write the field equations in the canonical form, that results in the evolution law of $\varphi^{(k)}$:

$$\mathrm{i}\partial_t \varphi^{(k)} = \delta H/\delta \varphi^{*(k)}. \tag{32}$$

Therefore, combining (19) and (32), one gets the evolution equation for Ψ_N:

$$\mathrm{i}\hbar\, \partial_t \Psi_N = \hbar \sum_{k=1}^{n} \sum_{j=1}^{N} \frac{\delta H}{\delta \varphi_j^{*(k)}} \frac{\partial \Psi_N}{\partial \varphi_j^{(k)}} \equiv \hat{H}\Psi_N, \tag{33}$$

which has the standard quantum mechanical form with some generalized Hamilton operator $\hat{H}$. As follows from (31), the operator $\hat{H}$ has the sense of the total energy operator of the system in question. Taking into account the estimate (24), one can ascertain that with the probability close to the unity the equation (33) is equivalent to some linear evolution equation for the probability amplitude [8].

Now we prove that in the nonrelativistic limit this equation should coincide with the Schrödinger equation for the system of n particles. In fact, according to (11) in the vicinity of the k–th particle the following equation holds:

$$\Box \varphi^{(k)} = (m_k c/\hbar)^2 \varphi^{(k)} + U_k(\phi, \pi),$$

which after the substitution

$$\varphi^{(k)} = u^{(k)} \mathrm{e}^{-\mathrm{i}m_k c^2 t/\hbar}$$

reduces, in the nonrelativistic limit, to the equation

$$\mathrm{i}\hbar\, \partial_t u^{(k)} \approx -\frac{\hbar^2}{2m_k} \triangle_k u^{(k)} + U_k',$$

where U_k' stands for an effective interaction potential. Therefore, the function

$$\psi_N = \Psi_N \exp\left(\sum_{k=1}^{n} \mathrm{i}m_k c^2 t/\hbar\right)$$

satisfies the standard n–particle Schrödinger equation.

Now it is worth-while to discuss the evidence of wave properties of particles in solitonian scheme. To verify the fact that solitons can really possess wave properties, the *gedanken* diffraction experiment with individual electrons—solitons was realized. Solitons with some velocity were thrown into a rectilinear slit, cut in the impermeable screen, and the transverse momentum was calculated which they gained while passing the slit, with the width of the latter significantly exceeded the size of the soliton. As a result, the picture of distribution of the centers of scattered solitons was restored on the registration screen, by considering their initial distribution to be uniform over the transverse coordinate. It was clarified that though the center of each soliton fell into a definite place of the registration screen (depending on the initial soliton profile and the point of crossing the plane of the slit by the soliton's center), the statistical picture in many ways was similar to the well-known diffraction distribution in optics, i.e. the Fresnel's picture at short distances from the slit and the Fraunhofer's one at large distances [15, 11].

Various aspects of the fulfillment of the quantum mechanics correspondence principle for the Einstein—de Broglie's solitonian model were discussed in the works [8, 9, 10]. In these papers it was shown that in the framework of the solitonian model all quantum postulates were regained in the limit of point-like particles, so that from the physical fields one can build the amplitude of probability and the mean value of some physical quantity can be calculated as a scalar product in the Hilbert space by introducing the corresponding quantum operators for observables. The fundamental role of the gravitational field in the de Broglie—Einstein solitonian scheme was discussed in [10, 16]. The solitonian model of the hydrogen atom was developed in [12, 13]. The dynamics of solitons in external fields was discussed in the paper [17].

As a result we obtain the stochastic realization (19) of the wave function Ψ_N which can be considered as an element of the random Hilbert space $\mathcal{H}_{\rm rand}$ with the scalar product

$$(\psi_1, \psi_2) = \mathbb{M}(\psi_1^* \psi_2), \tag{34}$$

with $\mathbb{M}$ standing for the expectation value. As a rude simplification one can admit that the averaging in (34) is taken over random characteristics of particles—solitons, such as their positions, velocities, phases, and so on. It is important to underline once more that the correspondence with the standard quantum mechanics is retained only in the point–particle limit ($\triangle \vee \gg \vee_0$) for $N \to \infty$. To show this [8, 9], one can apply the central limit theorem stating that for $N \to \infty$ the wave function $\Psi_N(t, \mathbf{x})$ behaves as the Gaussian random field with the variance

$$\sigma^2 = \rho(t, \mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^{3n}, \tag{35}$$

where $\rho(t, \mathbf{x})$ stands for the probability density (partition function) of solitons' centers in $\mathbb{R}^{3n}$.

Random Hilbert spaces being widely exploited in mathematical statistics [18], for quantum applications they were first used by N. Wiener in [19]. To illustrate the line of Wiener's argument, we recall the general scheme of introducing various representations in quantum mechanics.

Let $|\psi\rangle$ be a state vector in the Hilbert space $\mathcal{H}$ and $\hat{A}$ be a self-conjugate operator with the spectrum $\sigma(\hat{A})$. Then the a–representation is given by the wave function

$$\psi(a) = \langle a|\psi\rangle,$$

where

$$\hat{A}|a\rangle = a|a\rangle, \qquad a \in \sigma(\hat{A}).$$

In particular, the famous Schrödinger coordinate q–representation is given by the wave function

$$\psi(q) = \langle q|\psi\rangle = \sum_n \langle q|n\rangle\langle n|\psi\rangle, \tag{36}$$

with $|n\rangle$ being some complete set of state vectors in $\mathcal{H}$.

Wiener considered the real Brownian process $x(s, \alpha)$ in the interval $[0, 1] \ni s$, where $\alpha \in [0, 1]$ is the generalized number of the Brownian trajectory and the correlation reads

$$\int_0^1 {\rm d}\alpha\, x(s, \alpha) x(s', \alpha) = \min(s, s'). \tag{37}$$

To obtain the quantum mechanical description, Wiener defined the complex Brownian process

$$z(s|\alpha,\beta) = \frac{1}{\sqrt{2}}\,[x(s,\alpha) + \mathrm{i}\,y(s,\beta)]\,; \quad \alpha,\,\beta \in [0,1], \tag{38}$$

and using the natural mapping $\mathbb{R}^3 \to [0,1]$, for the particle in $\mathbb{R}^3$, constructed the stochastic representation of the wave function along similar lines as in (36):

$$\langle \alpha,\beta|\psi\rangle = \int_{s\in[0,1]} \mathrm{d}z(s|\alpha,\beta)\psi(s), \tag{39}$$

with the obvious unitarity property

$$\int_0^1 \mathrm{d}s\,|\psi(s)|^2 = \iint\limits_{[0,1]^2} \mathrm{d}\alpha\,\mathrm{d}\beta|\langle\alpha,\beta|\psi\rangle|^2$$

stemming from (37).

Entangled solitons and EPR correlations

In the sequel we shall consider the special case of two–particles configurations ($n = 2$), corresponding to the singlet state of two 1/2–spin particles. In quantum mechanics these states are described by the spin wave function of the form

$$\psi_{12} = \frac{1}{\sqrt{2}}\,(|1\uparrow\rangle \otimes |2\downarrow\rangle - |1\downarrow\rangle \otimes |2\uparrow\rangle) \tag{40}$$

and are known as **entangled states**. The arrows in (40) signify the projections of spin $\pm 1/2$ along some fixed direction. In the case of the electrons in the famous Stern—Gerlach experiment this direction is determined by that of an external magnetic field. If one chooses two different Stern—Gerlach devices, with the directions **a** and **b** of the magnetic fields, denoted by the unit vectors **a** and **b** respectively, one can measure the correlation of spins of the two electrons by projecting the spin of the first electron on **a** and the second one on **b**. Quantum mechanics gives for the spin correlation function the well-known expression

$$P(\mathbf{a},\,\mathbf{b}) = \psi_{12}^{+}(\sigma\mathbf{a}) \otimes (\sigma\mathbf{b})\psi_{12}, \tag{41}$$

where σ stands for the vector of Pauli matrices σ_i, $i = 1, 2, 3$. Putting (40) into (41), one easily gets

$$P(\mathbf{a},\,\mathbf{b}) = -(\mathbf{ab}). \tag{42}$$

The formula (42) characterizes the spin correlation in the Einstein—Podolsky—Rosen entangled singlet states and is known as the EPR–correlation. As was shown by J. Bell [20], the correlation (42) can be used as an efficient criterium for distinguishing the models with the local (point-like) hidden variables from those with the nonlocal ones. Namely, for the local-hidden-variables theories the EPR–correlation (42) is broken.

It would be interesting to check the solitonian model, shortly described in the beforehand points, by applying to it the EPR–correlation criterium. To this end let us first describe the 1/2–spin particles as solitons in the nonlinear spinor model of Heisenberg—Ivanenko type considered in the works [21, 22]. The soliton in question is described by the relativistic 4–spinor field φ of stationary type

$$\varphi = \begin{bmatrix} u \\ v \end{bmatrix} \mathrm{e}^{-\mathrm{i}\omega t}, \tag{43}$$

satisfying the equation

$$\left(\mathrm{i}\gamma^k\partial_k - \ell_0^{-1} + \lambda(\bar{\varphi}\varphi)\right)\varphi = 0, \tag{44}$$

where u and v denote 2–spinors, k runs Minkowsky space indices 0, 1, 2, 3; ℓ_0 stands for some characteristic length (the size of the particle—soliton), λ is self-coupling constant, $\bar{\varphi} \equiv \varphi^{+}\gamma^0$, γ^k are the Dirac matrices. The

stationary solution to the equation (44) can be obtained by separating variables in spherical coordinates r, ϑ, α via the substitution

$$u = \frac{1}{\sqrt{4\pi}} f(r) \left[\begin{array}{c} 1 \\ 0 \end{array} \right], \quad v = \frac{\mathrm{i}}{\sqrt{4\pi}} g(r) \sigma_r \left[\begin{array}{c} 1 \\ 0 \end{array} \right], \tag{45}$$

where $\sigma_r = (\sigma \mathbf{r})/r$. Inserting (45) into (44), one finds

$$\frac{\omega}{c} u + \mathrm{i}(\sigma \bigtriangledown) v - \ell_0^{-1} u + \frac{\lambda}{4\pi} \left(f^2 - g^2 \right) u = 0,$$
$$\frac{\omega}{c} v + \mathrm{i}(\sigma \bigtriangledown) u - \ell_0^{-1} v + \frac{\lambda}{4\pi} \left(f^2 - g^2 \right) v = 0.$$

In view of (45) one gets

$$\mathrm{i}(\sigma \bigtriangledown) v = -\frac{1}{\sqrt{4\pi}} \left(g' + \frac{2}{r} g \right) \left[\begin{array}{c} 1 \\ 0 \end{array} \right],$$
$$\mathrm{i}(\sigma \bigtriangledown) u = -\frac{\mathrm{i}}{\sqrt{4\pi}} f' \sigma_r \left[\begin{array}{c} 1 \\ 0 \end{array} \right].$$

Finally, one derives the following ordinary differential equations for the radial functions $f(r)$ and $g(r)$:

$$\left(g' + \frac{2}{r} g \right) = \left(\frac{\omega}{c} - \ell_0^{-1} \right) f + \frac{\lambda}{4\pi} \left(f^2 - g^2 \right) f,$$
$$-f' = \left(\frac{\omega}{c} + \ell_0^{-1} \right) g + \frac{\lambda}{4\pi} \left(f^2 - g^2 \right) g.$$

As was shown in the papers [21, 22], these equations admit regular solutions, if the frequency parameter ω belongs to the interval

$$0 < \omega < c/\ell_0. \tag{46}$$

The behavior of the functions $f(r)$ and $g(r)$ at $r \to 0$ is as follows:

$$g(r) = C_1 r, \quad f = C_2, \quad f' \to 0,$$

where C_1, C_2 denote some integration constants. The behavior of solutions far from the center of the soliton, i.e. at $r \to \infty$, is given by the relations:

$$f = \frac{A}{r} \mathrm{e}^{-\nu r}, \quad g = -\frac{f'}{B},$$

where

$$\nu = \left(\ell_0^{-2} - \omega^2/c^2 \right)^{1/2}, \quad B = \ell_0^{-1} + \omega/c.$$

If one chooses the free parameters ℓ_0 and λ of the model to satisfy the normalization condition (similar to (19))

$$\int \mathrm{d}^3 x \, \varphi^+ \varphi = \int_0^\infty \mathrm{d}r \, r^2 \left(f^2 + g^2 \right) = \hbar, \tag{47}$$

then the spin of the soliton reads

$$\mathbf{S} = \int \mathrm{d}^3 x \, \varphi^+ \mathbf{J} \varphi = \frac{\hbar}{2} \mathbf{e}_z, \tag{48}$$

where $\mathbf{e}_z$ denotes the unit vector along the Z–direction, $\mathbf{J}$ stands for the angular momentum operator

$$\mathbf{J} = -\mathrm{i}[\mathbf{r} \bigtriangledown] + \frac{1}{2} \sigma \otimes \sigma_0, \tag{49}$$

and σ_0 is the unit 2×2–matrix.

Now it is worth-while to show the positiveness of the energy E of the 1/2–spin soliton. The energy E is given by the expression

$$E = c \int \mathrm{d}^3x \left[-\mathrm{i}\varphi^+(\alpha\bigtriangledown)\varphi + \ell_0^{-1}\bar{\varphi}\varphi - \frac{\lambda}{2}(\bar{\varphi}\varphi)^2 \right], \tag{50}$$

where $\alpha = \sigma \otimes \sigma_1$. The positiveness of the functional (50) emerges from the virial identities characteristic for the model in question. In fact, the equation for the stationary solution (43) can be derived from the variational principle based on the Lagrangian of the system

$$L = -E + \int \mathrm{d}^3x\, \omega\varphi^+\varphi. \tag{51}$$

Performing the two-parameters scale transformation of the form $\varphi(x) \to \alpha\varphi(\beta x)$, one can derive from (51) and the variational principle $\delta L = 0$ the following two virial identities, which are valid for any regular stationary solution to the field equation (44):

$$\int \mathrm{d}^3x \left[\mathrm{i}\frac{2}{3}\varphi^+(\alpha\bigtriangledown)\varphi + \frac{\omega}{c}\varphi^+\varphi - \ell_0^{-1}\bar{\varphi}\varphi + \frac{\lambda}{2}(\bar{\varphi}\varphi)^2 \right] = 0, \tag{52}$$

$$\int \mathrm{d}^3x \left[\mathrm{i}\varphi^+(\alpha\bigtriangledown)\varphi + \frac{\omega}{c}\varphi^+\varphi - \ell_0^{-1}\bar{\varphi}\varphi + \lambda(\bar{\varphi}\varphi)^2 \right] = 0. \tag{53}$$

Using (52) and (53), one can express some sign-changing integrals through those of definite sign:

$$\int \mathrm{d}^3x \left[-\mathrm{i}\frac{1}{3}\varphi^+(\alpha\bigtriangledown)\varphi \right] = \frac{\lambda}{2} \int \mathrm{d}^3x\, (\bar{\varphi}\varphi)^2, \tag{54}$$

$$\int \mathrm{d}^3x \left[\ell_0^{-1}\bar{\varphi}\varphi + \frac{\lambda}{2}(\bar{\varphi}\varphi)^2 \right] = \frac{\omega}{c} \int \mathrm{d}^3x\, \varphi^+\varphi. \tag{55}$$

Using the identities (54) and (55), one can represent the energy (50) of the soliton as follows:

$$E = c \int \mathrm{d}^3x \left[\ell_0^{-1}\bar{\varphi}\varphi + \lambda(\bar{\varphi}\varphi)^2 \right] = \omega \int \mathrm{d}^3x\, \varphi^+\varphi = \hbar\omega, \tag{56}$$

where the normalization condition (47) was taken into account. Thus, one concludes, in the connection with (46) and (56), that the energy of the stationary spinor soliton (43) in the nonlinear model (44) turns out to be positive. Moreover, one can see that (56) is equivalent to the Planck—de Broglie wave—particle dualism relation (11).

Now let us construct the two–particles singlet configuration on the base of the soliton solution (43). First of all, in analogy with (40), one constructs the entangled solitons configuration endowed with the zero spin:

$$\varphi_{12} = \frac{1}{\sqrt{2}} \left[\varphi_1^{\uparrow} \otimes \varphi_2^{\downarrow} - \varphi_1^{\downarrow} \otimes \varphi_2^{\uparrow} \right], \tag{57}$$

where $\varphi_1^{\uparrow}$ corresponds to (45) with $\mathbf{r} = \mathbf{r}_1$, and $\varphi_2^{\downarrow}$ emerges from the above solution by the substitution

$$\mathbf{r}_1 \to \mathbf{r}_2, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

that corresponds to the opposite projection of spin on the Z–axis. In virtue of the orthogonality relation for the states with the opposite spin projections, one easily derives the following normalization condition for the entangled solitons configuration (57):

$$\int \mathrm{d}^3x_1 \int \mathrm{d}^3x_2\, \varphi_{12}^+\varphi_{12} = \hbar^2. \tag{58}$$

Now it is not difficult to find the expression for the stochastic wave function (20) for the singlet two–solitons state:

$$\Psi_N(t, \mathbf{r}_1, \mathbf{r}_2) = \left(\hbar^2 N\right)^{-1/2} \sum_{j=1}^{N} \varphi_{12}^{(j)}, \tag{59}$$

where $\varphi_{12}^{(j)}$ corresponds to the entangled soliton configuration in the j–th trial.

Our final step is the calculation of the spin correlation (41) for the singlet two–soliton state. In the light of the fact that the operator σ in (41) corresponds to the twice angular momentum operator (49), one should calculate the following expression:

$$P'(\mathbf{a},\mathbf{b}) = \mathbb{M} \int \mathrm{d}^3x_1 \int \mathrm{d}^3x_2\, \Psi_N^+ 2\,(\mathbf{J}_1\mathbf{a}) \otimes 2\,(\mathbf{J}_2\mathbf{b})\, \Psi_N, \tag{60}$$

where $\mathbb{M}$ stands for the averaging over the random phases of the solitons. Inserting (59) and (49) into (60), using the independence of trials $j \neq j'$ and taking into account the relations:

$$J_+\varphi^\uparrow = 0, \qquad J_3\varphi^\uparrow = \frac{1}{2}\varphi^\uparrow, \qquad J_-\varphi^\uparrow = \varphi^\downarrow,$$

$$J_-\varphi^\downarrow = 0, \qquad J_3\varphi^\downarrow = -\frac{1}{2}\varphi^\downarrow, \qquad J_+\varphi^\downarrow = \varphi^\uparrow,$$

where $J_\pm = J_1 \pm iJ_2$, one easily finds that

$$P'(\mathbf{a},\mathbf{b}) = -\hbar^{-2}\,(\mathbf{ab}) \left(\int_0^\infty \mathrm{d}r\, r^2 \left(f^2 + g^2\right) \right)^2 = -\,(\mathbf{ab})\,. \tag{61}$$

Comparing the correlations (61) and (42), one remarks their coincidence, that is the solitonian model satisfies the EPR–correlation criterium.

Conclusion

The main purpose of this paper was to find new arguments in favour of the thought that the soliton concept advocated by Einstein and de Broglie can give a consistent description of extended quantum particles. In particular, as a motivation for such a conclusion, within a framework of nonlinear spinor field model the solitonian image of 1/2–spin particles was used for constructing two–solitons singlet configuration, which permitted to calculate the spin EPR correlation. Fascinating result of this calculation was the coincidence of the quantum spin correlation with that in the solitonian scheme. This latter fact supports the hope that the solitonian scheme has many attractive features is relevant to consistent theory of extended elementary particles.

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Dynamics-generating semigroups and phenomenology of decoherence

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Abstract: *The earlier proposed Dynamics-Generating Approach (DGA) is reviewed and extended. Starting from an arbitrary chosen group or semigroup having structure similar to the structure of Galilei group, DGA allows one to construct phenomenological description of dynamics of the corresponding "elementary quantum object" (a particle or non-local object of special type). A class of Galilei-type semigroup, with semigroup of trajectories (parametrized paths) instead of translations, allows to derive, in the framework of DGA, Feynman path integrals. The measure of path integrating (exponential of the classic action) is not postulated but derived from the structure of projective semigroup representations. The generalization of DGA suggested in the present paper allows one to derive dynamics of open quantum systems. Specifically, phenomenological description of decoherence and dissipation of a non-relativistic particle is derived from Galilei semigroup.*

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1 Introduction

It is well-known that dynamics of a physical system is restricted if this system possesses some symmetry [1–3]. The formalism which is appropriate for describing symmetries, is group theory, and behavior of symmetric quantum systems is presented with the help of group representations.

It is less known that *dynamics* of elementary quantum objects (such as elementary particles but also some types of non-local objects) are not only restricted by symmetry of these objects but *can be derived from the group-theoretical considerations*. Dynamics may be derived from the given group or semigroup [4–6] in the approach that can be called *dynamics-generating approach*, or DGA.

The resulting dynamics depends on the choice of the group or semigroup that is a starting point of the procedure of DGA. The simplest choice is Galilei group, but in the general case the *dynamics-generating semigroup* (DGS) should have structure similar to the structure of Galilei group, in particularly has to include transformation similar to proper Galilei transformations or their generalizations. One may say that DGS should be a *Galiei-type group or semigroup*.

Taking *Galilei group* as DGS leads to theory of non-relativistic quantum particles [4]. Theory of relativistic particles follows [6] from the so-called *Aghassi-Roman-Santilly group* [7–9] which is a relativistic generalization of Galilei group, with the so-called proper time instead of the usual time.

Path integrals as the mathematical apparatus presenting quantum dynamics follows, in the framework of DGA, from the non-relativistic or relativistic *Galilei-type semigroup*, which includes the semigroup of parametrized paths (trajectories) instead of the usual translation group [5, 6]. This may result in theory of point particles, as in Feynman path integral theory. However, instead of this, one may obtain in this way a more general type of dynamics, corresponding non-local objects of special type. We call these objects *history-strings*, and they may be used as a sort of fundamental model for presenting confinement of quarks [13].

In the present paper we shall expand DGA in such a way that it might describe open quantum systems, so that the resulting dynamics might include decoherence and dissipation.

It seems unexpected that DGA may lead to phenomenological description of the dynamics of open quantum systems i.e. those which are subject to influence of their environment. However, this proves to be possible. Logics of the construction is based in this case on the fact that 1) decoherence and dissipation may be presented phenomenologically with the help of *restricted path integrals* (RPI), or *quantum corridors*, and 2) the dynamics presented by RPI may also be derived in the framework of DGA.

Dissipation presented by RPI
Decoherence and dissipation of a quantum system is a consequence of interaction of this system with its environment [11–14]. However, decoherence may in various ways be presented phenomenologically (see [15] and references therein). One of the phenomenological approaches is based on the idea that interaction with the environment remains in the latter some information on the state of the system and may therefore be considered as a continuous measurement of the system. One can make use of *restricted path integrals* (RPI), or *quantum corridors*, for describing such a measurement [5,16].

According to RPI approach, integration in Feynman path integral is restricted onto a set of paths (quantum corridor) which presents the readout of the continuous measurement (information remaining in the environment about the state of the system in all time moments). RPI presents therefore dynamics of an open system under action of its environment. However, no explicit model of the environment is used in the RPI approach.

Approach based on RPI is advantageous in case when a quantum system is given, and the task is in analyzing how the dynamics of this system change under action of various environments. It is much more cumbersome to apply in these situations conventional approach, with considering the system together with its environment and subsequent integrating out the degrees of freedom of the environment. The phenomenological RPI approach essentially simplifies calculations and allows to systematically explore large classes of environments for the given system.

The simplest model of decoherence and dissipation of a point quantum particle that can be obtained in the framework of RPI approach [15], leads to the phase representation of the path integral (the integral over paths in the phase space) in which the conventional measure of integration (exponential of the action) is substituted by the following functional:

$$\begin{aligned} U^t_{t'}([\mathbf{p}],[\mathbf{x}]) \;&=\; \exp\left\{\int_{t'}^{t} dt\left[\frac{i}{\hbar}\Big(\mathbf{p}\,\dot{\mathbf{x}} - H_0(\mathbf{p},\mathbf{x})\Big)\right.\right. \\ &\qquad \left.\left. -\kappa\Big(A(\mathbf{p},\mathbf{x}) - a(t)\Big)^2 - \frac{i}{\hbar}\Big(\lambda\, a(t) B(\mathbf{p},\mathbf{x}) + C(\mathbf{p},\mathbf{x})\Big)\right]\right\} \end{aligned}$$

The first term here (with $H_0 = \mathbf{p}^2/2m$) presents dynamics of the initial quantum system (point particle of mass m), the second term is responsible for decoherence arising due to continuous measuring the observable $\hat{A}$ (with the precision depending on the coefficient κ), resulting in the value $a(t)$ of this observable at time t, and the last term presents dissipation. More complicated regimes of decoherence may be described analogously, differing only by the choice of the corresponding functions in the exponent.

Semigroup of quantum corridors In the example presented above, the quantum corridor is determined by the function $[a]^t_{t'}$, i.e. by the valued $a(t)$ of the measured observable at each moment of the interval of the measurement. Operation of multiplication is naturally defined for these quantum corridors as $[a]^t_{t''} = [a]^t_{t'} \cdot [a]^{t'}_{t''}$ is taken. Two corridors that have to be multiplied are joined together in such a way that the second corridor starts at the point where the first one ends. With this definition, corridors form a semi-groupoid (since the inverse element is not defined for an arbitrary corridor and not any pair of the corridors may be multiplied).

However, the semi-groupoid can be converted into a semigroup (in which any pair of elements may be multiplied) if not individual corridors but classes of corridors are considered as elements, each class containing the corridors differing by the general shift of all its points. In the book [16] a group was additionally defined which transforms different corridors (i.e. different results of the continuous measurement) at the given time interval into one another. Finally, a semigroup of corridors may be defined that has a structure resembling the structure of Galilei group. Such Galilei-type semigroup can be used for deriving, in the framework of DGA, the dynamics of decohering or dissipating systems.

Here we shall obtain the dynamics of this type without postulating path integral but deriving it from the group-theoretical considerations formulated as dynamics-generating approach (DGA). The origin of the DGA process will be the choice of some Galilei-type semigroup (a group in simple case). All elements of dynamics, including the measure of path integrating (exponential of the classical action), will be determined from the chosen Galilei-type dynamics-generating semigroup (DGS).

2 Dynamics-generating approach (DGA)

In this section we shall briefly mention the main points of the dynamics-generating approach (DGA) leading from the arbitrarily chosen group or semigroup of Galilei type, G, to dynamics of elementary quantum objects (for simplicity, call them particles, although some of them may be non-local). The approach includes constructing two characteristic representations of G as well as intertwining these representations.

The *elementary representation* $U_{elem}(G)$ describes the particle as a whole (as an elementary object) while the *local representation* $U_{loc}(G)$ describes the same particle in terms of its localization in the appropriate space (for example space-time, but may be another space relating with the space-time in a more complicated way). Intertwining these representations (mapping between carrier spaces of them, $S_1 : \mathcal{L}_{elem} \to \mathcal{L}_{loc}$ and $S_2 : \mathcal{L}_{loc} \to \mathcal{L}_{elem}$) allows one to connect the two descriptions (elementary and local) with each other and derive the causal propagator, i.e. the probability amplitude for the particle to transit from one localized state to the other.

An important (although purely technical) role in this construction is played by induced representations of groups/semigroups that supply universal mathematical instruments for all stages of DGA. Remark that the group/semigroup should have a structure similar to the structure of Galilei group, but different Galilei-type groups/semigroups lead to different kinds of elementary objects and different kinds of dynamics, including even decoherence and dissipation.

2.1 Representations of dynamics-generating groups/semigroups

The local representation $U_{loc}(G)$ has to be realized in the space $\mathcal{L}_{loc}$ of functions on the configuration space of the particle. The latter may be the space-time for point particles, but can also have more complicated structure for non-local objects (or even for point particles but under external influence). In any case the configuration space may be realized as a quotient space G/H where $H \subset G$ is a subgroup/subsemigroup, and the local representation may be constructed as a representation $U_{loc}(G) = \chi(H) \uparrow G$ of the group or semigroup G induced from the subgroup/subsemigroup H (see below Sect. **??** 2.2sec:InducedRepresentations about induced representations).

The elementary representation $U_{elem}(G)$ should be irreducible (for presenting elementary object) and may be realized in any way. However, it is convenient to realize it also as an induced representation $U_{elem}(G) = \kappa(K) \uparrow G$ from an appropriate representation κ of an appropriate subgroup/subsemigroup $K \subset G$.

If the representations $U_{loc}(G)$ and $U_{elem}(G)$ are realized as induced representations, then the operators which intertwine these representations, are obtained in the standard form as it is shown in Sect. . The operators S_1 and S_2 are said to intertwine the representations $U_{elem}(G)$ and $U_{loc}(G)$ in one and opposite directions ($S_1 \in [U_{elem}, U_{loc}]$) and $S_2 \in [U_{loc}, U_{elem}]$) if the following commutation relations are valid:

$$S_1 U_{elem}(g) = U_{loc}(g) S_1, \quad S_2 U_{loc}(g) = U_{elem}(g) S_2.$$

These operators map the two representations onto each other, conserving the action of the group/semigroup. The product operator $\Pi = S_1 S_2$ intertwines the local representation $U_{loc}(G)$ with itself. The kernel $\Pi(x'', x')$ of the operator Π is then a two-point function with the arguments in the localization space (space-time in the simplest case). This function is interpreted as a probability amplitude for transition from the point x' in the localization space to another point x'' of this space. Causal propagator is obtained then by imposing additional condition that the propagation occurs from the past to the future.

This physical interpretation of the operator $\Pi = S_1 S_2$ is elaborated in the book [4] for point non-relativistic and relativistic particles (when localization space is the space-time) and applied in [5, 6] for deriving non-relativistic and relativistic path integrals (when the localization space is a space of trajectories).

Different choices of the dynamics-generating groups/semigroups lead to different dynamics:

- Galilei group: Non-relativistic particles
- Aghassi-Roman-Santilli group: Relativistic particles
- Galilei-type semigroup (with paths instead of tranlation vectors): Feynman path integrals

Different choices of configuration spaces describe different physical influences of environment on the particles:

- Space-time: Gauge fields (including confinement of color)
- Phase space: Decoherence and dissipation

Some of these constructions will be briefly considered in the next sections.

2.2 Induced representations

We shall use so-called induced representations of groups/semigroups as a convenient mathematical instrument. Induced representations generalize the well-known regular representation (acting by left shifts in the space of number-valued functions on the group). The representation $\kappa(K) \uparrow G$ of the group or semigroup G induced from the representation $\kappa(K)$ of the subgroup (subsemigroup) $K \subset G$ is also defined by left shifts of functions on G, but the space of these functions (a carrier space of $\kappa(K) \uparrow G$) is defined in a more complicated way.

The carrier space of the representation $U_\kappa(G) = \kappa(K) \uparrow G$ is defined as a space of functions on G with values in the carrier space $\mathcal{L}_\kappa$ of the representation $\kappa(K)$ and satisfying the so-called *structure condition*

$$\varphi(gk) = \kappa(k^{-1})\varphi(g) \tag{1}$$

(for any $g \in G$, $k \in K$. The operators of the induced representation U_κ act in this space by left shifts:

$$U_\kappa(g)\varphi(g') = \varphi(g^{-1}g') \tag{2}$$

Operators $U(g)$ are unitary in respect to the scalar product

$$(\varphi, \varphi') = \int_{G/K} \langle \varphi(x_G), \varphi'(x_G) \rangle dx \tag{3}$$

Intertwining $S \in [\kappa(K) \uparrow G, \chi(H) \uparrow G]$**:**
Operator which intertwines two induced representations of a group/semigroup G have form

$$(S\varphi)(g) = \int_G s(g')\varphi(gg')dg' \tag{4}$$

were operator-valued function $s(g) : \mathcal{L}_\kappa \to \mathcal{L}_\chi$ satisfies the following two-sided structure condition:

$$s(hgk) = \chi(h)s(g)\kappa(k), \quad \forall\, k \in K,\, h \in H$$

Because of this condition, the integrand is in fact constant on the cosets gK. Therefore, integration may be performed over the quotient space G/K rather than over the whole group/semigroup:

$$S\varphi(g) = \int_{G/K} s(x_G)\varphi(gx_G)dx \tag{5}$$

3 Galilei group and non-relativistic particles

The simplest case of applying DGA is the derivation of dynamics of non-relativistic particles from the Galilei group [4]. We shall briefly expose here the scheme of this construction. The more complicated applications in the following sections will follow the same general scheme.

3.1 Galilei group

Elements of Galilei group may be presented as products ($g = a_T r \mathbf{v}_L$) of translations, rotations and proper Galilei transformations. Each of these are defined by their action on the space-time points $x = \{x^0, \mathbf{x}\}$ as follows (notations are evident):

$$a_T x = x + a = \{x^0 + a^0, \mathbf{x} + \mathbf{a}\},$$
$$rx = \{x^0, r\mathbf{x}\}, \quad \mathbf{v}_G x = \{x^0, \mathbf{x} + x^0\mathbf{v}\}$$

Multiplications in Galilei group are completely determined by the following relations:

$$a_T a'_T = (a+a')_T, \quad \mathbf{v}_G \mathbf{v}'_G = (\mathbf{v}+\mathbf{v}')_G,$$
$$r\mathbf{v}_G r^{-1} = (r\mathbf{v})_G, \quad \mathbf{v}_G a_T \mathbf{v}_G^{-1} = (\mathbf{v}_G a)_T = \{a^0, \mathbf{a} + a^0\mathbf{v}\}_T$$

3.2 Projective representations as an origin of dynamics

Projective representations of the Galilei group are defined as

$$U(g)U(g') = \lambda(g,g')U(gg')$$

where $\lambda(g,g')$ are complex numbers called *multiplicators of the given representation*. The system of multiplicators for a projective representation of Galilei group is determined as

$$\lambda\big(a_T r\mathbf{v}_G, a'_T r'\mathbf{v}'_G\big) = \lambda\big(\mathbf{v}_G, a'_T\big) = \exp\left[im\left(\mathbf{v}\mathbf{a}' + \frac{1}{2}\mathbf{v}^2 a'^0\right)\right].$$

It depends on a single parameter m (which will play the role of mass in the dynamics resulting from DGA).

Propagator following from Galilei group due to DGA (i.e. as $\Pi = S_1 S_2$ with the operators S_i intertwining elementary and local representations) is equal (up to a number factor) [4] to

$$\begin{aligned}(\Pi\psi)(x) &= \int d^4x'\,\Pi(x-x')\psi(x'),\\ \Pi(x-x') &= \int d\mathbf{v}\,\exp\left\{i\left[m\mathbf{v}(\mathbf{x}-\mathbf{x}') - \frac{1}{2}m\mathbf{v}^2(x^0-x'^0)\right]\right\}.\end{aligned}$$

Causal propagator is obtained then if one requires that the transition is performed from the past to the future:

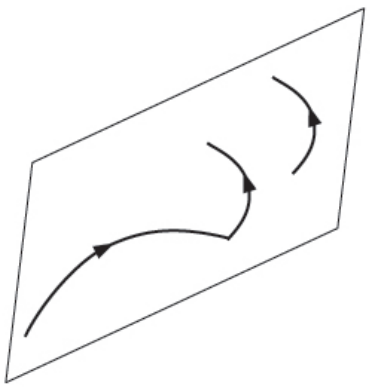
$$\Pi^c(x-x') = \theta(x^0-x'^0)\Pi(x-x').$$

4 Paths and Galilei semigroup

Galilei semigroup is obtained if translation subgroup of Galilei group is replaced by the semigroup of parametrized paths, or trajectories [7, 13, 16, 18]. Dynamics resulting in the framework of DGA [5] is what is known as Feynman path integrals. It will be considered in Sect. 5.1.

4.1 Paths instead of translations

The semigroup of trajectories (parametrized paths) is defined as a generalization of the translation group. This may be done as follows (see Fig. 1).



- An element of the translation group is defined as an element in the corresponding vector space. Product of two translations is defined as a sum of the corresponding vectors.
- Let us consider a generalized translation as a class of continuous curves (in the corresponding linear space), each class containing the curves obtained from each other by general shifts. Product of two classes is defined by prolongation of the representative of one class by the representative of the other class, taking the representatives in such a way that the end of one of them coincides with the starting point of the other.
- Precise definitions may be found in [7,13,16,18]. In the result, the set of generalized translations (classes of continuous curves) form a semigroup, i.e. associative product is defined for any pairs of elements but an inverse elements is not always defined.

If the paths are parametrized by the time parameter, we shall denote any curve $\mathbf{x}(t)$ belonging to this class as $[\mathbf{x}]_{t'}^{t''}$ where $\mathbf{x}(t')$ and $\mathbf{x}(t'')$ are its initial and final points. The path is defined as a class of equivalent curves, two curves being equivalent if all points of one of them differ from the points of the other curve by the same vector. We have thus the *semigroup of paths* (the term "path" denoting a class of equivalent curves). Because of the equivalence, the path, i.e. the class of equivalent curves, is defined by velocities $\dot{\mathbf{x}}(t)$. We shall denote such a class as $[\mathbf{x}]_{t'}^{t''} = [\dot{\mathbf{x}}]_{t'}^{t''}$ or even $[\mathbf{x}]_{t'}^{t''} = [\mathbf{u}]_{t''-t'}$ where $\mathbf{u}(t) = \dot{\mathbf{x}}(t-t')$.

4.2 Structure of Galilei semigroup

Galilei semigroup is defined if the translations of Galilei group are replaced by trajectories (or parametrized paths) $[\mathbf{x}]_{t'}^{t''} = [\mathbf{u}]_{t''-t'}$. Correspondingly, the elementary quantum object that will be described in the framework of DGA should have the space of trajectories as its localization space.

Generalization of a proper Galilei transformation is determined as family of velocities, one velocity for each time: $[\mathbf{v}] = \{\mathbf{v}(t) | -\infty < t < \infty\}$ with products defined trivially:

$$[\mathbf{v}][\mathbf{v}'] = [\mathbf{v}+\mathbf{v}'] = \{\mathbf{v}(t)+\mathbf{v}'(t) | -\infty < t < \infty\}. \tag{6}$$

Besides the mentioned elements, Galilei semigroup contains rotations $r \in R$, with the evident relations:

$$r[\mathbf{v}]r^{-1} = [r\mathbf{v}], \quad r[\mathbf{u}]_\tau r^{-1} = [r\mathbf{u}]_\tau, \tag{7}$$

The Galilei semigroup as a whole contains these elements as well as their products, $g = [\mathbf{u}]_\tau r[\mathbf{v}]$.

Interrelations of the characteristic elements of Galilei semigroup may be characterized as follows. As has been said previously, product of paths (trajectories) $[\mathbf{x}]_{t'}^{t''}[\mathbf{x}]_{t}^{t'}$ is defined as prolonging (extending) them by each other. The action of a proper Galilei transformation $[\mathbf{v}]$ converts the paths on the given time interval into each other, which is expressed by the following commutation relation:

$$[\mathbf{v}][\mathbf{u}]_\tau[\mathbf{v}]^{-1} = [\mathbf{u}+\mathbf{v}]_\tau. \tag{8}$$

The Galilei semigroup (just as Galilei group) has (non-trivial) projective representations, with the systems of multiplicators of the form

$$\left([\mathbf{u}]_\tau r[\mathbf{v}], [\mathbf{u}']_\tau r'[\mathbf{v}']\right) = \exp\left[im\int_0^\tau d\sigma\left(\mathbf{u}\mathbf{v}' + \frac{1}{2}\mathbf{v}'^2\right)\right]. \tag{9}$$

Instead of projective representations of Galilei semigroup one can consider only usual vector representation but for central extension of the semigroup (we shall denote it by the same letter G). Elements of the extended semigroup are defined as $g = \lambda[\mathbf{u}]_\tau r[\mathbf{v}]$ where λ is an arbitrary complex number (therefore, an element from the center of the semigroup G). For elements of the extended semigroup the same relations are valid but with Eq. (8) replaced by the following:

$$[\mathbf{v}][\mathbf{u}]_\tau[\mathbf{v}]^{-1} \quad = \quad \lambda\cdot[\mathbf{u}+\mathbf{v}]_\tau, \quad \lambda = \exp\left[im\int_0^\tau d\sigma\left(\mathbf{u}\mathbf{v} + \frac{1}{2}\mathbf{v}^2\right)\right]$$

5 Propagators in DGA with paths instead of translations

If we start the procedure of DGA from the Galilei semigroup, we obtain finally [5] the dynamics presented by Feynman path integrals. The concept of path integrals, including the measure of integrating (the famous exponential of the classical action) is not postulated in this case but derived from the group-theoretical considerations.

In the context of DGA, Galilei semigroup leads to the space of trajectories as a localization space. In principle, this allows one to obtain the dynamics of non-local objects.[1] If our goal is theory of point particles, then we make use of the propagator in the space of trajectories (Sect. 5.1) and go over to the propagator in the space-time (as is shown in Sect. 5.2).

5.1 Causal propagator in the space of trajectories

In Sect. 2.1 the general procedure (DGA) is exposed for constructing causal propagator of the elementary quantum object with the help of intertwining local and elementary representations of the dynamics-generating group/semigroup. This procedure consists in i) constructing intertwining operators $S_1 \in [U_{elem}, U_{loc}]$, $S_2 \in [U_{loc}, U_{elem}]$, and $\Pi = S_1 S_2$, ii) finding the kernel $\Pi(x'', x')$ of the latter and iii) imposing the causality condition for obtaining the causal propagator $\Pi^c(x'', x')$.

In case of Galilei semigroup, with paths instead of space-time translations, the role of localization space is played by trajectories (parametrized paths) [5,6]. The resulting form of the causal propagator, up to a number factor, is then [5]

$$(\Pi^c \psi)[\mathbf{x}]_0^t = \theta(t-t') \int_0^t dt' \int d[\mathbf{p}]_{t'}^t \cdot U_{t'}^t([\mathbf{p}],[\mathbf{x}]) \cdot \psi[\mathbf{x}]_0^{t'} \tag{10}$$

where it is denoted

$$U_{t'}^t([\mathbf{p}],[\mathbf{x}]) = \exp\left[\frac{i}{\hbar}\int_{t'}^t dt\,(\mathbf{p}\dot{\mathbf{x}} - H_0(\mathbf{p}))\right] \tag{11}$$

with

$$H_0(\mathbf{p}) = \frac{1}{2m}\mathbf{p}^2$$

This form of the propagator is formally derived from intertwining the local and elementary representations. We see that the Feynman measure in the space of trajectories is not postulated (as an exponential of the classical action) but derived from the group-theoretical considerations, namely from the system of multiplicators (9) of Galilei semigroup.

5.2 Propagator in space-time

In Sect. 5.1 we considered the space of trajectories (parametrized paths) as a localization space of our elementary quantum object. Therefore, this object may be in principle non-local. Such non-local objects (called 'history-strings') were considered in [6,13]. However, dynamics of a point particle can also be derived from the path-integral formalism obtained in Sect. 5.1.

For this aim, we have to define the probability amplitude to be in a definite space-time point as a sum of the amplitudes to arrive to this point along various paths (trajectories).

The action of paths on the space-time may be naturally defined as

$$(\mathbf{x}'', t'') = [\mathbf{x}]_{t'}^{t''}(\mathbf{x}', t')$$

where $(\mathbf{x}', t')$ and $(\mathbf{x}'', t'')$ are correspondingly the initial and final space-time points of the path $[\mathbf{x}]_{t'}^{t''}$. Let us say in this case that the point $(\mathbf{x}'', t'')$ is arrived from $(\mathbf{x}', t')$ along the path $[\mathbf{x}]_{t'}^{t''}$.

Let us suppose[2] that 1) at the time moment $t = 0$ the point particle is localized in a single point $(\mathbf{x}_0, 0)$ and 2) it can arrive to an arbitrary space-time points $(\mathbf{x}', t')$ along paths $[\mathbf{x}]_0^{t'}$ according to the formula

$$(\mathbf{x}', t') = [\mathbf{x}]_0^{t'}(\mathbf{x_0}, 0).$$

[1] Such non-local objects can naturally present confinement of quarks [13].

[2] This assumption may be generalized, but this is not essential for us now.

Let the wave function of the particle in the space of trajectories is $\psi[\mathbf{x}]_{t''}^{t'''}$. This means that the probability amplitude for the particle to move along the path $[\mathbf{x}]_{t''}^{t'''}$ is equal to $\psi[\mathbf{x}]_{t''}^{t'''}$. Then the amplitude for this particle to arrive to the point $(\mathbf{x}', t')$ along the path $[\mathbf{x}]_0^{t'}$ is equal to $\psi[\mathbf{x}]_0^{t'}$. Therefore, the amplitude to be at this point (i.e. to arrive to this point along any path leading to it) is equal to

$$\Psi(\mathbf{x}', t') = \int_{\mathbf{x}''}^{\mathbf{x}'} d[\mathbf{x}]_0^{t'} \alpha[\mathbf{x}]_0^{t'} \psi[\mathbf{x}]_0^{t'}, \quad \text{where} \quad (\mathbf{x}', t') = [\mathbf{x}]_0^{t'}(\mathbf{x}_0, 0) \tag{12}$$

with some weight function $\alpha[\mathbf{x}]_0^{t'}$. This is nothing else than a formula for transition from the wave function (of a point particle) given as a function on the space of trajectories, to the wave function on the space-time.

The weight function $\alpha[\mathbf{x}]_0^{t'}$ is not arbitrary. In order for the formula (12) to be in accord with the multiplicative structure of the paths, the following condition should be valid,

$$\alpha[\mathbf{x}]_{t'}^{t} \cdot \alpha[\mathbf{x}]_{t''}^{t'} = \alpha[\mathbf{x}]_{t''}^{t}, \quad \text{where} \quad [\mathbf{x}]_{t'}^{t} \cdot [\mathbf{x}]_{t''}^{t'} = [\mathbf{x}]_{t''}^{t}.$$

In other words, the function α has to be a representation of the semigroup of trajectories. It is easy to show [5] that the most general form of the weight function with these properties is

$$\alpha\{\mathbf{x}\}_{t''}^{t'} = T \exp\left\{ i \int_{t''}^{t'} dt \; [V(\mathbf{x}(t), t) + \mathbf{A}(\mathbf{x}(t), t)\dot{\mathbf{x}}] \right\}.$$

where "$T\exp$" denotes time-ordered exponential, V a potential and $\mathbf{A}$ a gauge field.

Equation (10) described causal propagation in terms of path-dependent wave function. Going over, for a point particle, to the point-dependent wave function (12), we have the point-dependent form of the causal propagator [5]

$$\begin{aligned} (\Pi^c \Psi)(\mathbf{x}, t) \quad &= \quad \theta(t - t') \int_0^t dt' \int d\tilde{\mathbf{x}} \int_{\tilde{\mathbf{x}}}^{\mathbf{x}} d[\mathbf{x}]_{t'}^{t} \cdot \alpha[\mathbf{x}]_{t'}^{t} \int d[\mathbf{p}]_{t'}^{t} \\ &\quad \times \exp\left\{ \int_{t'}^{t} dt \left[\frac{i}{\hbar} (\mathbf{p}\dot{\mathbf{x}} - H_0(\mathbf{p}) \right] \right\} \cdot \Psi(\tilde{\mathbf{x}}, t'). \end{aligned}$$

It coincides with the usual path-integral form of non-relativistic causal propagator.

6 Phase space and decoherence in DGA

Dynamics-generating approach (DGA) is aimed at the derivation of dynamics of elementary quantum objects, for example elementary particles. It seems at first glance that dynamics of complicated physical systems hardly can be derived in the same way. Even more strange to believe that dynamics of open quantum systems can be obtained in the framework of DGA.

The system is called open if its interaction with the environment affects its dynamics. As a result of this influence, the open quantum system exhibits the phenomenon called decoherence. In the course of gradual decoherence, the quantum system partially loses the quantum character of its evolution, behaves more like a classical system. Dissipation is an extreme form of decoherence.

It is unexpected that dynamics of open quantum systems, including phenomena of decoherence and dissipation, also can be derived in the framework of DGA. From the mathematical point of view, the price paid for this, is necessity to consider the (generalized) phase space as a localization space for our "elementary quantum object" (although this term becomes questionable in this situation).

In fact, this is not so strange because open quantum systems can be described phenomenologically, so that the environment is not explicitly included in the description (but its affect is taken into account implicitly). Here we shall remind the phenomenological description of decoherence by restricted path integrals (RPI) [5, 16] and then derive the same dynamics in the framework of DGA.

6.1 Decoherence in RPI approach

Decoherence as an effect of the environment can be taken into account directly in models similar to the widely known Caldeira-Leggett model [19], or with the help of the Feynman-Vernon influence functional included in the path integral [20]. Instead of this, one may use the phenomenological Lindblad equation [21]. We shall prefer to make use of the RPI approach (see [5,16] on continuous quantum measurements in terms of RPI and [15] on its application to decoherence and dissipation).

The role of environment may be considered to be a sort of continuous measurement, and dynamics of continuously measured quantum system may be accounted by restricting its path integral onto the corresponding *quantum corridor*, i.e. on the family of paths which corresponds to the result (output) of the measurement. This gives [15] for the propagator of the open (continuously measured) system the expression

$$U_\alpha(q'',q') = \int_{q'}^{q''} d[p]d[q]\cdot W_\alpha[p,q]\cdot\exp\left\{\frac{i}{\hbar}\int_t^{t'} dt\,(p\dot{q}-H(p,q,t))\right\}$$

where α denotes the measurement result and the weight functional $W_\alpha[p,q]$ restricts Feynman path integral on the subset of paths corresponding to the measurement result α.

In [15] the author considered the special case of a non-relativistic particle moving through the medium. This may be interpreted as continuous measuring some observable $\hat{A}$ of the particle, with the result of the measurement given by the values $a(t)$ of this observable in various time moments, i.e. by the curve $\alpha=[a]_{t'}^{t''}=\{a(t)|t'<t<t''\}$. This is described by the restricted path integral for the non-relativistic particle, for example in the form Eq. (10) but with a more complicated kernel $U_{t'}^{t''}$:

$$\begin{aligned} U_{t'}^{t''}([\mathbf{p}],[\mathbf{x}]) \;=\; & \exp\left\{\int_{t'}^{t''} dt\left[\frac{i}{\hbar}\Big(\mathbf{p}\,\dot{\mathbf{x}}-H_0(\mathbf{p},\mathbf{x})\Big)\right.\right. \\ & \left.\left.-\kappa\Big(A(\mathbf{p},\mathbf{x})-a(t)\Big)^2-\frac{i}{\hbar}\Big(\lambda\,a(t)B(\mathbf{p},\mathbf{x})+C(\mathbf{p},\mathbf{x})\Big)\right]\right\} \end{aligned} \tag{13}$$

The term $-\kappa(A-a)^2$ here describes the restriction of the path integral that corresponds to the continuous measurement of A with the precision determined by the coefficient κ. This term is responsible for decoherence. The terms including B and C, take into account an additional effect of dissipation that can arise in the process of the measurement.[3] For example the terms of this type may take into account deceleration by measurement, i.e. dissipation of energy.

This phenomenological description of decoherence and dissipation has been obtained in terms of restricted path integral. This type of dynamics emerges as a secondary effect rather than is defined on the fundamental level. It is questionable therefore whether it can be derived in the framework of DGA. We shall see in Sect. that this is possible if one introduce the (generalized) phase space as a localization space of our "elementary quantum object".

6.2 Propagator in phase space

Propagator in the space of paths is defined by Eq. (10) with the kernel (11), i.e. in the following form:

$$(\Pi^c\psi)[\mathbf{x}]_0^t=\int_0^t dt'\int d[\mathbf{p}]_{t'}^t\,\exp\left\{\int_{t'}^t dt\left[\frac{i}{\hbar}\Big(\mathbf{p}\,\dot{\mathbf{x}}-H_0(\mathbf{p})\Big)\right]\right\}\psi[\mathbf{x}]_0^{t'}$$

We have seen in Sect. 5.2 how the propagator in the space-time can be obtained from this general path-dependent formula. Let us now apply the same path-dependent propagator for deriving *propagator in the (generalized) phase space* and further for derivation phenomenological description of decoherence and dissipation.

The construction in Sect. 5.2 began by transition from the path-dependent wave function to the space-time-dependent wave function (12). Now we have to go over *from the space of paths to the generalized phase space* rather than to space-time.

[3] The form of the "decoherence term" $-\kappa(A-a)^2$ in the exponent of Eq. (13) is defined by the very general arguments [22], but additional terms may have a more complicated form.

The point of the usual phase space is characterized by the pair "position and momentum". We shall denote the point of the generalized phase space as the pair $\Big([\mathbf{p}]_0^{t'};(\mathbf{x}',t')\Big)$ including a space-time point and a path in the momentum space. The wave function in this space has to be obtained with the help of the procedure similar to one applied in Sect. 5.2. The value of the wave function in the definite point $\Big([\mathbf{p}]_0^{t'};(\mathbf{x}',t')\Big)$ of the generalized phase space is equal to the integral over all paths leading to this point. This results in the wave function of the form

$$\Psi\Big([\mathbf{p}]_0^{t'};(\mathbf{x}',t')\Big)=\int_{\mathbf{x}''}^{\mathbf{x}'}d[\mathbf{x}]_0^{t'}\,\alpha([\mathbf{p}],[\mathbf{x}])_0^{t'}\,\psi[\mathbf{x}]_0^{t'}$$

where α is some weight function depending on the paths in the phase space.[4]

The propagator in the generalized phase space is then derived along the arguments similar to those from Sect. 5.2:

$$\begin{aligned}\left(\Pi^c\Psi\right)\left([\mathbf{p}]_0^t;(\mathbf{x},t)\right) &= \int_0^t dt'\int d[\mathbf{p}]_{t'}^t\int d[\mathbf{x}]_0^t\\ &\times\ \alpha_0^t([\mathbf{p}],[\mathbf{x}])\cdot\exp\left\{\int_{t'}^t dt\left[\frac{i}{\hbar}\Big(\mathbf{p}\,\dot{\mathbf{x}}-H_0(\mathbf{p})\Big)\right]\right\}\\ \Psi\Big([\mathbf{p}]_0^{t'};(\mathbf{x}',t')\Big)&\end{aligned}$$

or

$$\left(\Pi^c\Psi\right)\left([\mathbf{p}]_0^t;(\mathbf{x},t)\right)=\int_0^t dt'\int d[\mathbf{p}]_{t'}^t\int d[\mathbf{x}]_0^t\cdot U_{t'}^t([\mathbf{p}],[\mathbf{x}])\,\Psi\Big([\mathbf{p}]_0^{t'};(\mathbf{x}',t')\Big)$$

where it is denoted

$$U_{t'}^t([\mathbf{p}],[\mathbf{x}])=\alpha_0^t([\mathbf{p}],[\mathbf{x}])\cdot\exp\left\{\int_{t'}^t dt\left[\frac{i}{\hbar}\Big(\mathbf{p}\,\dot{\mathbf{x}}-H_0(\mathbf{p})\Big)\right]\right\}. \tag{14}$$

Depending on the choice of the weight function $\alpha_0^t([\mathbf{p}],[\mathbf{x}])$, Eq. (14) presents general form of decoherence and dissipation of the non-relativistic particle. In particular, the kernel describing decoherence and dissipation of the form

$$\begin{aligned}U_{t'}^t([\mathbf{p}],[\mathbf{x}]) &= \exp\left\{\int_{t'}^t dt\left[\frac{i}{\hbar}\Big(\mathbf{p}\,\dot{\mathbf{x}}-H_0(\mathbf{p},\mathbf{x})\Big)\right.\right.\\ &\left.\left.-\kappa\Big(A(\mathbf{p},\mathbf{x})-a(t)\Big)^2-\frac{i}{\hbar}\Big(\lambda\,a(t)B(\mathbf{p},\mathbf{x})+C(\mathbf{p},\mathbf{x})\Big)\right]\right\}\end{aligned}$$

(as in Eq. (13)) may be obtained if α is chosen as follows:

$$\alpha\left([\mathbf{x}]_{t'}^t,[\mathbf{p}]_{t'}^t\right)=$$
$$\exp\left\{-\int_{t'}^t dt\left[\frac{i}{\hbar}\Big(+\lambda\,a(t)B(\mathbf{p},\mathbf{x})+C(\mathbf{p},\mathbf{x})\Big)+\kappa\Big(A(\mathbf{p},\mathbf{x})-a(t)\Big)^2\right]\right\}$$

After integrating in $[\mathbf{p}]_0^t$, we have finally the causal propagator in space-time

$$\begin{aligned}\left(\Pi^c\Psi\right)(\mathbf{x},t)=\textstyle\int_0^t dt'\int d\tilde{\mathbf{x}}\int_{\tilde{\mathbf{x}}}^{\mathbf{x}}d[\mathbf{x}]_{t'}^t\int d[\mathbf{p}]_{t'}^t\\ \times\exp\left\{\textstyle\int_{t'}^t dt\left[\frac{i}{\hbar}\Big(\mathbf{p}\,\dot{\mathbf{x}}-H_0(\mathbf{p},\mathbf{x})-C(\mathbf{p},\mathbf{x})-\lambda\,a(t)B(\mathbf{p},\mathbf{x})\Big)\right.\right.\\ \left.\left.-\kappa\Big(A(\mathbf{p},\mathbf{x})-a(t)\Big)^2\right]\right\}\cdot\Psi(\tilde{\mathbf{x}},t')\end{aligned}$$

[4] This weight function implicitly accounts for the influence of the environment and therefore determines details of dynamics of the open system.

correctly describing decoherence and dissipation,[5] i.e. influence of the environment. However, we derived the propagator in the framework of the universal procedure of dynamics-generating approach. The price for inclusion decoherence and dissipation in the scope of this approach is that we came to the propagation in space-time not directly, but through the intermediate form of the propagator in the generalized phase space.

7 Concluding remarks

In the preceding we gave a brief review of the Dynamics-Generating Approach (DGA) and demonstrated that it is applicable for derivating dynamics of open quantum systems.

The goal of DGA is to derive phenomenological description of dynamics of an "elementary quantum object" (called "particle" although it can be non-local). DGA starts from choosing some group or semigroup G having structure similar to the structure of Galilei group. Then two special representations of this group/semigroup are constructed, $U_{elem}(G)$ presenting the state of the particle as a whole, and $U_{loc}(G)$ describing localization of this state. Intertwining these representations allows one to provide agreement of these two description and derive propagator of the particle in an appropriate localization space (it may be space-time or the space of paths in the space-time).

It has been earlier shown that, depending on the concrete choice of the dynamics-generating semigroup G, the dynamics of a non-relativistic or relativistic local or non-local object can be obtained in the framework of DGA. Feynman path integral is derived in such a way, and the measure of the path integrating (exponential of the classical action) is also derived from G rather than postulated.

It is additionally shown in the present paper that dynamics of open quantum systems, including phenomena of decoherence and dissipation, can also be derived in such a way. This is achieved if the localization space of the "particle" is defined as a generalized phase space and propagation in this space is first derived as an intermediate point of the theory. In such a way the phenomenological description of a non-relativistic particles subjected to decoherence and dissipation is derived. Although this type of dynamics physically emerges as an affect of the environment, in the framework of DGA the environment is not considered explicitly. The decohering particle is considered in this case as a special sort of an elementary quantum object.

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Gravitational repulsion instead of event horizon

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Abstract: *Extended radial carriers of mass-energy obey Newtonian attraction in weak fields but repulsion in strong ones. Space interference of overlapping radial particles results in dipole, dark mass-energy which strictly balance negative gravitational potentials of material space continuum. The Newton fall is followed by the strong-field repulsion next to the gravitational center that rejects the black hole horizon, explains the Hubble expansion speed rH_o with calculated acceleration $r(H_o)^2$, and complies with Penrose's cyclic cosmology.*

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Schwarzschild's metric solution has no mathematical errors for the empty-space paradigm where elementary sources are point particles or Dirac delta-functions. In 1939 Einstein finally rejected Schwarzschild's metric with singularities for physical reality [1]. Indeed, metric solutions without singularities do exist in Einstein's General Relativity if the latter accepts the non-empty space paradigm or continuous sources-particles, for example [2,3]. New physics of geometrized radial particles describes main relativistic tests more self-consistently [4] than the point particle physics.

Overlapping continuous masses with local energy exchanges can justify a dark kind of mechanical mass-energy called interference or dipole energy. The many-body metric solution for static non-empty space [3] employs the post-Newton gravitational potential of overlapping radial masses

$$W(\mathbf{x}) \equiv -c^2 ln \frac{1}{\sqrt{g_{oo}(\mathbf{x})}} = -c^2 ln\left(1+\frac{r_1}{|\mathbf{x}-\mathbf{a}_1|}+\frac{r_2}{|\mathbf{x}-\mathbf{a}_2|}+...+\frac{r_n}{|\mathbf{x}-\mathbf{a}_n|}\right). \quad (1)$$

Here $r_i \equiv GE_i/c^4 = Gm_i/c^2$ is Schwarzschild-type coordinate scale of the elementary energy-charge E_i, distributed everywhere but mainly in the vicinity of $\mathbf{a}_i$. Once "absurd" Newtonian ether or nowadays material space with such a local metric stress $W(\mathbf{x})$ complies with the Einstein Principle of Equivalence for mechanical (inertial or passive, $\mu_p c^2$) and gravitational (potential or active, $\mu_a c^2$) mass-energy densities,

$$\mu_p(\mathbf{x}) \equiv \frac{[\nabla W(\mathbf{x})]^2}{4\pi G c^2} = \frac{\nabla^2 W(\mathbf{x})}{4\pi G} \equiv \mu_a(\mathbf{x}). \quad (2)$$

The logarithmic local potential (1) keeps integrals of active (potential) and passive (inertial) metric space energies, $\int d^3x \mu_p c^2 = \int d^3x \mu_a c^2 = E_{metric}$, of the united material continuum of n overlapping energy-charges:

$$E_{metric} \equiv \frac{c^4}{4\pi G}\int d^3x \left(\frac{\frac{(\mathbf{x}-\mathbf{a}_1)r_1}{|\mathbf{x}-\mathbf{a}_1|^3}+\frac{(\mathbf{x}-\mathbf{a}_2)r_2}{|\mathbf{x}-\mathbf{a}_2|^3}+...+\frac{(\mathbf{x}-\mathbf{a}_n)r_n}{|\mathbf{x}-\mathbf{a}_n|^3}}{1+\frac{r_1}{|\mathbf{x}-\mathbf{a}_1|}+\frac{r_2}{|\mathbf{x}-\mathbf{a}_2|}+...+\frac{r_n}{|\mathbf{x}-\mathbf{a}_n|}}\right)^2 = (m_1+m_2+...+m_n)c^2 = const. \quad (3)$$

Such a universal mass-energy conservation, $E_{metric} \equiv E_{monopoles} + E_{dipoles} = const$, for a system of interacting continuous particles can take place due to hidden energy contributions into paired (dipole, interference) formations of material densities, with $E_{metric} \equiv E_{monopoles} + E_{dipoles} = const$.

Despite negative potential energy shifts take place for observable radial elements, metric mass of the many-body system in (3) stays steady, $\sum m_i = const$, due to compensating deposits from dipole positive energies. This keeps scalar masses of many-particle bodies in spite of mutual internal interactions of material elements. When centers of radial particles are separated into finite distances, then n-body metric energy $c^2 \sum m_i$ contains both

directly observable (radial) and non-observable (dipole, dark) fractions of gravitational/inertial mass-energy. It should be used $|\mathbf{a}_k-\mathbf{a}_i|\equiv R_{ik}>>r_i+r_k=G(m_i+m_k)/c^2$ for distances between centers of radial particles in (3) at the most of weak-field applications,

$$E_{monopoles}\approx\frac{c^4}{4\pi G}\int d^3x\frac{r_1^2}{|\mathbf{x}|^4(1+\frac{r_1}{|\mathbf{x}|}+\frac{r_2}{|\mathbf{a}_1-\mathbf{a}_2|}+...+\frac{r_n}{|\mathbf{a}_1-\mathbf{a}_n|})^2}+\frac{c^4}{4\pi G}\int d^3x\frac{r_2^2}{|\mathbf{x}|^4(1+\frac{r_1}{|\mathbf{a}_2-\mathbf{a}_1|}+\frac{r_2}{|\mathbf{x}|}+...+\frac{r_n}{|\mathbf{a}_2-\mathbf{a}_n|})^2}+...+\frac{c^4}{4\pi G}\int d^3x\frac{r_n^2}{|\mathbf{x}|^4(1+\frac{r_1}{|\mathbf{a}_n-\mathbf{a}_1|}+\frac{r_2}{|\mathbf{a}_n-\mathbf{a}_2|}+...+\frac{r_n}{|\mathbf{x}|})^2}=c^2\sum_{i=1}^{n}m_i\sqrt{g_{oo}^{\neq i}(\mathbf{a}_i)}\approx\sum_{i=1}^{n}m_i\left(c^2-\sum_{k\neq i}^{n}\frac{Gm_k}{R_{ik}}\right)>0 \tag{4}$$

Here items for static monopole mass-energies, like $E_2\equiv c^2m_2\sqrt{g_{oo}^{\neq 2}(\mathbf{a}_2)}\equiv c^2m_2/\left(1+\frac{r_1}{|\mathbf{a}_2-\mathbf{a}_1|}+\frac{r_3}{|\mathbf{a}_2-\mathbf{a}_3|}+...+\frac{r_n}{|\mathbf{a}_2-\mathbf{a}_n|}\right)$, for example, contain negative shifts, associated with paired Newtonian interactions within the united material space. Negative Newtonian potentials for energy of monopoles (4) do not mean decrease of the system metric energy (3), because paired gravitational interactions are always accompanied by interference, dark deposits in a form of dipole energy formations,

$$E_{dipoles}=\frac{c^4}{4\pi G}\sum_{i=1}^{n}\sum_{k\neq i}^{n}\int_0^{2\pi}d\varphi\int_0^{\infty}r^2dr\int_0^{\pi}\frac{r_ir_k\left(r^2-R_{ik}rcos\theta\right)sin\theta d\theta}{r^3(R_{ik}^2+r^2-2R_{ik}rcos\theta)^{3/2}}\approx\sum_{i=1}^{n}\sum_{k\neq i}^{n}\frac{Gm_im_k}{R_{ik}}>0. \tag{5}$$

Gravitational attractions of positive energy bodies are always accompanied in (3)-(4) by positive energy of interference (dipole) fields. In fact, gravitation is not a formal decrease of negative potential energy of Newtonian field (which without host radial particles does not exist in (4) as a self-maintained field), but the universal tendency of a free mechanical system toward distribution of its total energy between all physical degrees of freedom. Equipartition distributions of mechanical energy between observable monopoles and dark dipoles may be expected, in principle, for an equilibrium gravitational system.

Material space continuum in Einstein's GR metric formalism always keeps Euclidean 3D section of curved 4D space-time due to inherent symmetries [2] of the real world geometry. GR geodesic equations of motions in pseudo-Riemann space-time with $0\leq g_{oo}\leq 1$ and flat 3D intervals, $g_{oi}g_{oj}g_{oo}^{-1}-g_{ij}=\delta_{ij}$, have been derived [4] for strong static fields,

$$\begin{cases} g_{oo}dt/dp=1, dp/ds=g_{oo}dt/ds=E_m/m=\text{const} \\ r^2d\varphi/dp=J_\varphi=\text{const}, r^2d\varphi/ds=J_\varphi E_m/m\equiv L=\text{const} \\ (dr/dp)^2+(J_\varphi/r)^2-g_{oo}^{-1}=\text{const}\ (=-m^2/E_m^2) \\ (dr/ds)^2+(rd\varphi/ds)^2-E_m^2/m^2g_{oo}=-1, \end{cases} \tag{6}$$

where $m=\text{const}$ is the probe scalar mass, while energy $E_m=\text{const}$ and angular momentum $J_\varphi=\text{const}$ are the first integrals of relativistic motion. For the pure radial fall from infinity, when $E_m/m\equiv c^2\sqrt{g_{oo}}/\sqrt{1-v^2c^{-2}}=const\Rightarrow 1$ and $d\varphi/ds=0$, $ds=\sqrt{g_{oo}}cdt\sqrt{1-v^2c^{-2}}, v^2=(dr/\sqrt{g_{oo}}dt)^2$, the last equation in (6) results in

$$dr/dt=\pm c\sqrt{g_{oo}(1-g_{oo})} \tag{7}$$

for the free radial motion with respect to the world (coordinate) time t of a distant observer. Static metric field (4) with one gravitating center, $g_{oo} = 1/[1 + (r_o/r)]^2$, leads in (7) to (unstable) motionless states, $dr/dt = 0$, of small probe masses at final stages of their radial falls. A probe mass reaches maximum radial speed $dr/dt = c/2$ of the central field fall at $r = r_o(1 + \sqrt{2})$. Below this transition distance the decelerating part of the fall takes place due to gravitational repulsion of strong fields.

Coordinate acceleration d^2r/dt^2 can be derived from (7) by taking its time derivative,

$$d^2\mathbf{r}/dt^2 = -c^2 r_o \mathbf{r}(r^2 - 2r_o r - r_o^2)/(r + r_o)^5. \tag{8}$$

This relation universally describes the Newton attraction $-r_o c^2\mathbf{r}/r^3$ for $r_o \ll r$ and the strong-field GR repulsion $+\mathbf{r}c^2/r_o^2$ for $r \ll r_o \equiv GM/c^2$. According to the metric stress presentation (4), both repulsion and attraction of free probe masses correspond to their motion in always negative gravitational potentials.

Today the radial dimension of the Metagalaxy is less than its gravitational scale $R_o = GM_{Meta}/c^2$. Therefore, such a dense Metagalaxy should repeal its material elements behind $R_o(1 + \sqrt{2})$. The strong-field limit of (7), when $r \ll r_o$ and $dr/dt = cr/R_o \Rightarrow rH_o$, corresponds to the Hubble expansion speed at $R_o \Rightarrow c/H_o = 1,3 \times 10^{26} m$ or at $M_{Meta} = R_o c^2/G = 1.8 \times 10^{53} kg$. The Universe expansion acceleration in this limit, $d^2r/dt^2 = c^2 r/R_o^2 \Rightarrow rH_o^2$, is proportional to the distance r like the Hubble expansion rate.

The Big Bang fragmentation of a radial monopole into the system of expanding (with acceleration) radial monopoles and dark dipoles corresponds to (7)-(8) and to the aforementioned tendency to equipartition distribution of energy between monopole and dipole degrees of freedom. In this way, all Metagalaxy's matter in whole is provisionally in the phase of strong-field expansion with acceleration. One day a mature Universe with constant metric mass-energy (5) of its continuous material space will enter into the contraction phase toward its configuration equilibrium next to the equipartition distribution of dark and observable energy contents within the united material space. The global Universe pendulum around equilibrium material densities of its nonempty space corresponds to the Penrose's basic construction for conformal cycling cosmology [5].

In general, dynamics of the pulsating metric space should count kinetic energy items for the equipartition distribution of energy between mechanical degrees of freedom of monopoles and dipoles. Such a dynamical systems with gravitational repulsion and attraction next to the equilibrium radius $r_o(1+\sqrt{2})$ can be applied, in particular, to a super massive center of a galaxy. In this way, the strong field gravitational repulsion is enabling proper readings of Sagittarius A* images at the center of the Milky Way Galaxy without event horizon options required by the black hole approach.

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Geometric Spinors, Generalized Dirac Equation and Mirror Particles

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Abstract: *It is shown that since the geometric spinors are elements of Clifford algebras, they must have the same transformation properties as any other Clifford number. In general, a Clifford number Φ transforms into a new Clifford number ϕ' according to $\Phi \to \Phi' = \mathrm{R}\,\Phi\,\mathrm{S}$, i.e., by the multiplication from the left and from the right by two Clifford numbers R and S. We study the case of $Cl(1,3)$, which is the Clifford algebra of the Minkowski spacetime. Depending on choice of R and S, there are various possibilities, including the transformations of vectors into 3-vectors, and the transformations of the spinors of one minimal left ideal of $Cl(1,3)$ into another minimal left ideal. This, among others, has implications for understanding the observed non-conservation of parity.*

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Introduction

We will follow the approach [1]– [4] in which spinors are constructed in terms of nilpotents formed from the spacetime basis vectors represented as generators of the Clifford algebra

$$\begin{aligned} \gamma_a \cdot \gamma_b &\equiv \tfrac{1}{2}\left(\gamma_a\gamma_b + \gamma_b\gamma_a\right) = \eta_{ab} \\ \gamma_a \wedge \gamma_b &\equiv \tfrac{1}{2}\left(\gamma_a\gamma_b - \gamma_b\gamma_a\right). \end{aligned} \tag{1}$$

The inner, symmetric, product of basis vectors gives the metric. The outer, antisymmetric, product of basis vectors gives a basis bivector.

The generic Clifford number is

$$\Phi = \varphi^A \gamma_A. \tag{2}$$

where $\gamma_A \equiv \gamma_{a_1 a_2 \ldots a_r} \equiv \gamma_{a_1} \wedge \gamma_{a_2} \wedge \ldots \wedge \gamma_{a_r}, \quad r = 0, 1, 2, 3, 4.$

Spinors are particular Clifford numbers, $\Psi = \psi^\alpha \xi_\alpha$, where ξ_α are spinor basis elements, composed from γ_A. We will consider transformation properties of Clifford numbers.

In general, a Clifford number transforms according to

$$\Phi \to \Phi' = \mathrm{R}\,\Phi\,\mathrm{S}. \tag{3}$$

Here R and S are Clifford numbers, e.g., $\mathrm{R} = \mathrm{e}^{\frac{1}{2}\alpha^A \gamma_A}$, $\mathrm{S} = \mathrm{e}^{\frac{1}{2}\beta^A \gamma_A}$.

In particular, if $S = 1$, we have

$$\Phi \to \Phi' = \mathrm{R}\,\Phi. \tag{4}$$

As an example, let us consider the case

$$\mathrm{R} = \mathrm{e}^{\frac{1}{2}\alpha\gamma_1\gamma_2} = \cos\frac{\alpha}{2} + \gamma_1\gamma_2 \sin\frac{\alpha}{2}\,, \qquad \mathrm{S} = \mathrm{e}^{\frac{1}{2}\beta\gamma_1\gamma_2} = \cos\frac{\beta}{2} + \gamma_1\gamma_2 \sin\frac{\beta}{2} \tag{5}$$

and examine [5], how various Clifford numbers,

$$X = X^C \gamma_C, \tag{6}$$

transform under (3), which now reads:

$$X \to X' = \mathrm{R}\,X\,\mathrm{S}. \tag{7}$$

(i) If $X = X^1\gamma_1 + X^2\gamma_2$ then

$$X' = X^1\left(\gamma_1\cos\frac{\alpha-\beta}{2} + \gamma_2\sin\frac{\alpha-\beta}{2}\right) + X^2\left(-\gamma_1\sin\frac{\alpha-\beta}{2} + \gamma_2\cos\frac{\alpha-\beta}{2}\right). \tag{8}$$

(ii) If $X = X^3\gamma_3 + X^{123}\gamma_{123}$ then

$$X' = X^3\left(\gamma_3\cos\frac{\alpha+\beta}{2} + \gamma_{123}\sin\frac{\alpha+\beta}{2}\right) + X^{123}\left(-\gamma_3\sin\frac{\alpha+\beta}{2} + \gamma_{123}\cos\frac{\alpha+\beta}{2}\right). \tag{9}$$

(iii) If $X = s\,\underline{1} + X^{12}\gamma_{12}$, then

$$X' = s\left(\underline{1}\cos\frac{\alpha+\beta}{2} + \gamma_{12}\sin\frac{\alpha+\beta}{2}\right) + X^{12}\left(-\underline{1}\sin\frac{\alpha+\beta}{2} + \gamma_{12}\cos\frac{\alpha+\beta}{2}\right). \tag{10}$$

(iv) If $X = \tilde{X}^1\gamma_5\gamma_1 + \tilde{X}^2\gamma_5\gamma_2$, then

$$X' = \tilde{X}^1\left(\gamma_5\gamma_1\cos\frac{\alpha-\beta}{2} + \gamma_5\gamma_2\sin\frac{\alpha-\beta}{2}\right) + \tilde{X}^2\left(-\gamma_5\gamma_1\sin\frac{\alpha-\beta}{2} + \gamma_5\gamma_2\cos\frac{\alpha-\beta}{2}\right). \tag{11}$$

Usual rotations of vectors or pseudovectors are reproduced, if the angle β for the right transformation is equal to minus angle for the left transformation, i.e., if $\beta = -\alpha$. Then all other transformations which mix the grade vanish. But in general, if $\beta \neq \alpha$, the transformation (7) mixes the grade.

Clifford algebra and spinors in Minkowski space

Let us introduce a new basis, called the Witt basis,

$$\begin{array}{l}\theta_1 = \frac{1}{2}(\gamma_0+\gamma_3)\,, \theta_2 = \frac{1}{2}(\gamma_1+i\gamma_2)\,,\\ \bar{\theta}_1 = \frac{1}{2}(\gamma_0-\gamma_3)\,, \bar{\theta}_2 = \frac{1}{2}(\gamma_1-i\gamma_2),\end{array} \tag{12}$$

where

$$\gamma_a = (\gamma_0,\gamma_1,\gamma_2,\gamma_3). \tag{13}$$

The new basis vectors satisfy

$$\{\theta_a,\bar{\theta}_b\} = \eta_{ab}, \quad \{\theta_a,\theta_b\} = 0, \quad \{\bar{\theta}_a,\bar{\theta}_b\} = 0, \tag{14}$$

which are fermionic anticommutation relations. We now observe that the product

$$f = \bar{\theta}_1\bar{\theta}_2 \tag{15}$$

satisfies

$$\bar{\theta}_a\, f = 0\,, \qquad a = 1,2. \tag{16}$$

Here f can be interpreted as a 'vacuum', and $\bar{\theta}_a$ can be interpreted as operators that annihilate f.

An object constructed as a superposition

$$\Psi = (\psi^0 + \psi^1\theta_1 + \psi^2\theta_2 + \psi^{12}\theta_1\theta_2)f \tag{17}$$

is a 4-component spinor. It is convenient to change the notation:

$$\Psi = (\psi^1 + \psi^2\theta_1\theta_2 + \psi^3\theta_1 + \psi^4\theta_2)f \; = \psi^\alpha \xi_\alpha \,, \qquad \alpha = 1,2,3,4 \tag{18}$$

where ξ_α is the spinor basis.

The even part of the above expression is a left handed spinor

$$\Psi_L = (\psi^1 + \psi^2\theta_1\theta_2)\,\bar\theta_1\bar\theta_2, \tag{19}$$

whereas the odd part is a right handed spinor

$$\Psi_R = (\psi^3\theta_1 + \psi^4\theta_2)\bar\theta_1\bar\theta_2. \tag{20}$$

We can verify that the following relations are satisfied:

$$i\gamma_5\Psi_L = -\Psi_L, \qquad i\gamma_5\Psi_R = \Psi_R \tag{21}$$

Under the transformations

$$\Psi \to \Psi' = \mathrm{R}\Psi \,, \tag{22}$$

where

$$\mathrm{R} = \exp[\frac{1}{2}\gamma_{a_1}\gamma_{a_2}\varphi], \tag{23}$$

the Clifford number Ψ transforms as a spinor.

As an example let us consider the case

$$\mathrm{R} = \mathrm{e}^{\frac{1}{2}\gamma_1\gamma_2\varphi} = \cos\frac{\varphi}{2} + \gamma_1\gamma_2 \sin\frac{\varphi}{2}. \tag{24}$$

Then we have

$$\Psi \to \Psi' = \mathrm{R}\Psi \; = \left(\mathrm{e}^{\frac{i\phi}{2}}\psi^1 + \mathrm{e}^{-\frac{i\phi}{2}}\psi^2\theta_1\theta_2 + \mathrm{e}^{\frac{i\phi}{2}}\psi^3\theta_1 + \mathrm{e}^{-\frac{i\phi}{2}}\psi^4\theta_2\right) f. \tag{25}$$

This is the well-known transformation of a 4-component spinor.

Four independent spinors

There exist four different possible vacua [3, 4, 6]:

$$f_1 = \bar\theta_1\bar\theta_2 \,, \qquad f_2 = \theta_1\theta_2 \,, \qquad f_3 = \theta_1\bar\theta_2 \,, \quad f_3 = \bar\theta_1\theta_2 \tag{26}$$

to which there correspond four different kinds of spinors:

$$\begin{array}{l} \Psi^1 = (\psi^{11} + \psi^{21}\theta_1\theta_2 + \psi^{31}\theta_1 + \psi^{41}\theta_2)f_1 \\ \Psi^2 = (\psi^{12} + \psi^{22}\bar\theta_1\bar\theta_2 + \psi^{32}\bar\theta_1 + \psi^{42}\bar\theta_2)f_2 \\ \Psi^3 = (\psi^{13}\bar\theta_1 + \psi^{23}\theta_2 + \psi^{33} + \psi^{43}\bar\theta_1\theta_2)f_3 \\ \Psi^4 = (\psi^{14}\theta_1 + \psi^{24}\bar\theta_2 + \psi^{34} + \psi^{44}\theta_1\bar\theta_2)f_4. \end{array} \tag{27}$$

Each of those spinors lives in a different minimal left ideal of $Cl(1,3)$, or in general, of its complexified version if we assume complex $\psi^{\alpha i}$.

An arbitrary element of $Cl(1,3)$ is the sum:

$$\Phi = \Psi^1 + \Psi^2 + \Psi^3 + \Psi^4 = \psi^{\alpha i}\xi_{\alpha i} \equiv \psi^{\tilde A}\xi_{\tilde A}, \tag{28}$$

where

$$\xi_{\tilde{A}} \equiv \xi_{\alpha i} = \{f_1,\ \theta_1\theta_2 f_1, ..., \theta_1 f_4,\ \bar{\theta}_2 f_4,\ f_4,\ \bar{\theta}_1\theta_2 f_4\}, \tag{29}$$

is a spinor basis of $Cl(1,3)$. Here Φ is a generalized spinor.

In matrix notation we have

$$\psi^{\alpha i} = \begin{pmatrix} \psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\ \psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\ \psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\ \psi^{41} & \psi^{42} & \psi^{43} & \psi^{44} \end{pmatrix},$$

$$\xi_{\tilde{A}} \equiv \xi_{\alpha i} = \begin{pmatrix} f_1 & f_2 & \bar{\theta}_1 f_3 & \theta_1 f_4 \\ \theta_1\theta_2 f_1 & \bar{\theta}_1\bar{\theta}_2 f_2 & \theta_2 f_3 & \bar{\theta}_2 f_4 \\ \theta_1 f_1 & \bar{\theta}_1 f_2 & f_3 & f_4 \\ \theta_2 f_1 & \bar{\theta}_2 f_2 & \bar{\theta}_1\theta_2 f_3 & \theta_1\bar{\theta}_2 f_4 \end{pmatrix}. \tag{30}$$

Here, for instance, the second column in the left matrix contains the components of the spinor of the second left ideal. Similarly, the second column in the right matrix contains the basis elements of the second left ideal.

A general transformation is

$$\Phi = \psi^{\tilde{A}}\xi_{\tilde{A}} \to \Phi' = \mathrm{R}\,\Phi\,\mathrm{S} = \psi^{\tilde{A}}\xi'_{\tilde{A}} = \psi^{A} L_{\tilde{A}}{}^{\tilde{B}}\xi_B = \psi'^{\tilde{B}}\xi_{\tilde{B}} \tag{31}$$

where

$$\xi'_{\tilde{A}} = \mathrm{R}\xi_{\tilde{A}}\mathrm{S} = L_{\tilde{A}}{}^{\tilde{B}}\xi_{\tilde{B}}\,, \qquad \psi'^{\tilde{B}} = \psi^{\tilde{A}} L_{\tilde{A}}{}^{\tilde{B}}. \tag{32}$$

This is an active transformation, because it changes an object Φ into another object Φ'.

The transformation from the left,

$$\Phi' = \mathrm{R}\,\Phi, \tag{33}$$

reshuffles the components within each left ideal, whereas the transformation from the right,

$$\Phi' = \Phi\,\mathrm{S}, \tag{34}$$

reshuffles the left ideals.

Behavior of spinors under Lorentz transformations

Let us consider the following transformation of the basis vectors

$$\gamma_a \to \gamma'_a = \mathrm{R}\,\gamma_a \mathrm{R}^{-1}\,, \qquad a = 0, 1, 2, 3, \tag{35}$$

where R is a proper or improper Lorentz transformation. A generalized spinor, $\Phi \in Cl(1,3)$, composed of γ_a, then transforms according to

$$\Phi = \psi^{\tilde{A}}\xi_{\tilde{A}} \to \Phi' = \psi^{\tilde{A}}\xi'_{\tilde{A}} = \psi^{A}\mathrm{R}\,\xi_B\mathrm{R}^{-1} = \mathrm{R}\,\Phi\,\mathrm{R}^{-1}. \tag{36}$$

The transformation (35) of the basis vectors has for a consequence that the object Φ does not transform only from the right, but also from the left. This had led Piazzese to the conclusion that spinors cannot be interpreted as the minimal ideals of Clifford algebras [7].

But if the reference frame transforms as

$$\gamma_a \to \gamma'_a = \mathrm{R}\,\gamma_a, \tag{37}$$

then

$$\Phi = \psi^{\tilde{A}}\xi_{\tilde{A}} \to \Phi' = \psi^{\tilde{A}}\xi'_{\tilde{A}} = \psi^{\tilde{A}}\mathrm{R}\,\xi_{\tilde{B}} = \mathrm{R}\,\Phi. \tag{38}$$

This is a transformation of a spinor. Therefore, the description of spinors in terms of ideals is consistent.

As we have seen in Sec.1, the transformation (37) is also a possible transformation within a Clifford algebra. It is a transformation that changes the grade of a basis element. Usually, we do not consider such transformations of basis vectors. Usually reference frames are "rotated" (Lorentz rotated) according to

$$\gamma_a \to \gamma'_a = \mathrm{R}\,\gamma_a \mathrm{R}^{-1} = L_a{}^b \gamma_b, \tag{39}$$

where $L_a{}^b$ is a proper or improper Lorentz transformation. Therefore, a "rotated" observer sees (generalized) spinors transformed according to

$$\Phi \to \Phi' = \mathrm{R}\,\Phi\,\mathrm{R}^{-1}. \tag{40}$$

With respect to a new reference frame, the object $\Phi = \psi^{\tilde{A}} \xi_{\tilde{A}}$ is expanded according to

$$\Phi = \psi'^{\tilde{A}} \xi'_{\tilde{A}}, \tag{41}$$

where

$$\psi'^{\tilde{A}} = \psi^{\tilde{B}} (L^{-1})_{\tilde{B}}{}^{\tilde{A}}. \tag{42}$$

Recall that $\alpha, \beta = 1, 2, 3, 4$, and $i, j = 1, 2, 3, 4$. The corresponding matrix $\psi^{\alpha i}$ transforms from the left and from the right.

If the observer, together with the reference frame, starts to rotate, then after having exhibited the $\varphi = 2\pi$ turn, he observes the same spinor Ψ, as he did at $\varphi = 0$. The sign of the spinor did not change, because this was just a passive transformation, so that the same (unchanged) objects was observed from the transformed (rotated) references frames at different angles φ. In the new reference frame the object was observed to be transformed according to $\Psi' = R\Psi R^{-1}$. There must also exist the corresponding *active* transformation such that in a fixed reference frame the spinor transforms as $\Psi' = R\Psi R^{-1}$.

Examples

Rotation

Let us consider the following rotation:

$$\begin{aligned} &\gamma_0 \to \gamma_0\,, \quad \gamma_1 \to \gamma_1\,, \quad \gamma_2 \to \gamma_2 \cos\vartheta + \gamma_3 \sin\vartheta \\ &\qquad\qquad\qquad\quad \gamma_3 \to -\gamma_2 \sin\vartheta + \gamma_3 \cos\vartheta. \end{aligned} \tag{43}$$

In the case $\vartheta = \pi$, we have

$$\gamma_0 \to \gamma_0\,, \quad \gamma_1 \to \gamma_1\,, \quad \gamma_2 \to -\gamma_2\,, \quad \gamma_3 \to -\gamma_3. \tag{44}$$

The Witt basis then transforms as

$$\theta_1 \to \bar{\theta}_1\,, \quad \theta_2 \to \bar{\theta}_2\,, \quad \bar{\theta}_1 \to \theta_1\,, \quad \bar{\theta}_2 \to \theta_2. \tag{45}$$

A consequence is that, e.g., a spinor of the first left ideal transforms as

$$(\psi^{11} + \psi^{21}\theta_1\theta_2 + \psi^{31}\theta_1 + \psi^{41}\theta_2)\,\bar{\theta}_1\bar{\theta}_2 \;\to (\psi^{11} + \psi^{21}\bar{\theta}_1\bar{\theta}_2 + \psi^{31}\bar{\theta}_1 + \psi^{41}\bar{\theta}_2)\,\theta_1\theta_2. \tag{46}$$

By inspecting the latter relation and taking into account Eqs.(27), (19),(20), we see that a left handed spinor of the *first ideal* transforms into a left handed spinor of the *second ideal*. Similarly, a right handed spinor of the first ideal transforms into a right handed spinor of the second ideal.

In general, under the $\vartheta = \pi$ rotation in the (γ_2, γ_3) plane, a generalized spinor

$$\begin{aligned} \Phi \;=\; &(\psi^{11} + \psi^{21}\theta_1\theta_2 + \psi^{31}\theta_1 + \psi^{41}\theta_2)\bar{\theta}_1\bar{\theta}_2 \\ &+(\psi^{12} + \psi^{22}\bar{\theta}_1\bar{\theta}_2 + \psi^{32}\bar{\theta}_1 + \psi^{42}\bar{\theta}_2)\theta_1\theta_2 \\ &+(\psi^{13}\bar{\theta}_1 + \psi^{23}\theta_2 + \psi^{33} + \psi^{43}\bar{\theta}_1\theta_2)\theta_1\bar{\theta}_2 \\ &+(\psi^{14}\theta_1 + \psi^{24}\bar{\theta}_2 + \psi^{34} + \psi^{44}\theta_1\bar{\theta}_2)\bar{\theta}_1\theta_2 \end{aligned} \tag{47}$$

transforms into

$$\begin{array}{rcl}\Phi' &=& (\psi^{11}+\psi^{21}\bar\theta_1\bar\theta_2+\psi^{31}\bar\theta_1+\psi^{41}\bar\theta_2)\theta_1\theta_2\\ &&+(\psi^{12}+\psi^{22}\theta_1\theta_2+\psi^{32}\theta_1+\psi^{42}\theta_2)\bar\theta_1\bar\theta_2\\ &&+(\psi^{13}\theta_1+\psi^{23}\bar\theta_2+\psi^{33}+\psi^{43}\theta_1\bar\theta_2)\bar\theta_1\theta_2\\ &&+(\psi^{14}\bar\theta_1+\psi^{24}\theta_2+\psi^{34}+\psi^{44}\bar\theta_1\theta_2)\theta_1\bar\theta_2\end{array}\quad . \tag{48}$$

The matrix of components

$$\psi^{\alpha i}=\begin{pmatrix}\psi^{11}&\psi^{12}&\psi^{13}&\psi^{14}\\ \psi^{21}&\psi^{22}&\psi^{23}&\psi^{24}\\ \psi^{31}&\psi^{32}&\psi^{33}&\psi^{34}\\ \psi^{41}&\psi^{42}&\psi^{43}&\psi^{44}\end{pmatrix}$$

transforms into

$$\psi'^{\alpha i}=\begin{pmatrix}\psi^{12}&\psi^{11}&\psi^{14}&\psi^{13}\\ \psi^{22}&\psi^{21}&\psi^{24}&\psi^{23}\\ \psi^{32}&\psi^{31}&\psi^{34}&\psi^{33}\\ \psi^{42}&\psi^{41}&\psi^{44}&\psi^{43}\end{pmatrix}. \tag{49}$$

We see that in the transformed matrix, the first and the second column are interchanged. Similarly, also the third and forth column are interchanged. Different columns represent different left minimal ideals of $Cl(1,3)$, and thus different spinors.

Let us now focus our attention on the spinor basis states of the first and second ideal:

$$\xi_{11}=\bar\theta_1\bar\theta_2\,,\quad \xi_{21}=\theta_1\theta_2\bar\theta_1\bar\theta_2\,,\quad \xi_{12}=\theta_1\theta_2\,,\quad \xi_{22}=\bar\theta_1\bar\theta_2\theta_1\theta_2. \tag{50}$$

which span the left handed part of the 4-component
spinor (see Eqs.(19,20)).

Under the $\vartheta=\pi$ rotation (44), (45), we have

$$\xi_{11}\to\xi_{12}\,,\quad \xi_{21}\to\xi_{22}\,,\quad \xi_{12}\to\xi_{11}\,,\quad \xi_{22}\to\xi_{21}, \tag{51}$$

which means that the spin $1/2$ state of the 1st ideal transforms into the spin state of the 2nd ideal, and vice versa. The above states are eigenvalues of the spin operator, $-\frac{i}{2}\,\gamma_1\gamma_2$,

$$-\frac{i}{2}\gamma_1\gamma_2\,\xi_{11}=\frac{1}{2}\xi_{11}\,,\quad ,\quad -\frac{i}{2}\gamma_1\gamma_2\,\xi_{21}=-\frac{1}{2}\xi_{21}\,, \tag{52}$$

$$-\frac{i}{2}\gamma_1\gamma_2\,\xi_{12}=-\frac{1}{2}\xi_{12}\,,\quad -\frac{i}{2}\gamma_1\gamma_2\,\xi_{22}=\frac{1}{2}\xi_{22}\,. \tag{53}$$

Let us now introduce the new basis states

$$\begin{array}{l}\xi^1_{1/2}=\frac{1}{\sqrt2}(\xi_{11}+\xi_{22})\,,\quad \xi^2_{1/2}=\frac{1}{\sqrt2}(\xi_{11}-\xi_{22})\,,\\ \xi^1_{-1/2}=\frac{1}{\sqrt2}(\xi_{21}+\xi_{12})\,,\ ,\ \xi^2_{-1/2}=\frac{1}{\sqrt2}(\xi_{21}-\xi_{12})\,.\end{array} \tag{54}$$

which are superpositions of the states of the 1st and the 2nd ideal. Under the rotation (44),(45) we have

$$\begin{array}{l}\xi^1_{1/2}\to\frac{1}{\sqrt2}(\xi_{12}+\xi_{21})=\xi^1_{-1/2}\,,\\ \xi^1_{-1/2}\to\frac{1}{\sqrt2}(\xi_{22}+\xi_{11})=\xi^1_{1/2}\,,\end{array} \tag{55}$$

$$\begin{array}{l}\xi^2_{1/2}\to\frac{1}{\sqrt2}(\xi_{12}-\xi_{21})=-\xi^2_{-1/2},\\ \xi^2_{-1/2}\to\frac{1}{\sqrt2}(\xi_{22}-\xi_{11})=-\xi^2_{1/2}\,.\end{array} \tag{56}$$

These states also have definite spin projection:

$$-\frac{i}{2}\gamma_1\gamma_2\xi^1_{\pm 1/2} = \pm\frac{1}{2}\xi^1_{\pm 1/2}\,, \tag{57}$$

$$-\frac{i}{2}\gamma_1\gamma_2\xi^2_{\pm 1/2} = \pm\frac{1}{2}\xi^2_{\pm 1/2}\,. \tag{58}$$

The states (55) have the property that under the $\vartheta = \pi$ rotation, the spin $1/2$ state $\xi^1_{1/2}$ transforms into the spin $-1/2$ state $\xi^1_{-1/2}$, and vice versa. Analogous hold for the other set of states, $\xi^2_{1/2}$, $\xi^2_{-1/2}$.

Let us stress again that the transformation in the above example is of the type $\Phi' = R\Phi R^{-1}$. This is a reason that, under such a transformation, a spinor of one ideal is transformed into the spinor of a different ideal. A transformation R^{-1}, acting from the right, mixes the ideals. Another kind of transformation is $\Phi' = R\Phi$, in which case there is no mixing of ideals. Such are the usual transformations of spinors. By considering the objects of the entire Clifford algebra and possible transformations among them, we find out that spinors are not a sort of objects that transform differently than vectors under rotations. They can transform under rotations in the same way as vectors, i.e., according to $\Phi' = R\Phi R^{-1}$. Here Φ can be a vector, spinor or any other object of Clifford algebra. In addition to this kind of transformations, there exist also the other kind of transformations, namely, $\Phi' = R\Phi$, where again Φ can be any object of $Cl(1,3)$, including a vector or a spinor. These are particular cases of the more general transformations, $\Phi' = R\Phi S$, considered in Sec. 1.

Space inversion

Let us now consider space inversion, under which the basis vectors of a reference frame transform according to

$$\gamma_0 \to \gamma'_0 = \gamma_0\,, \quad \gamma_r \to \gamma'_r = -\gamma_r\,, \quad r = 1,2,3\,. \tag{59}$$

The vectors of the Witt basis (12) then transform as

$$\begin{array}{l} \theta_1 \to \frac{1}{2}(\gamma_0 - \gamma_3) = \bar\theta_1\,, \\ \theta_2 \to \frac{1}{2}(-\gamma_1 - i\gamma_2) = -\theta_2, \\ \bar\theta_1 \to \frac{1}{2}(\gamma_0 + \gamma_3) = \theta_1\,, \\ \bar\theta_2 \to \frac{1}{2}(-\gamma_1 + i\gamma_2) = -\bar\theta_2\,. \end{array}\qquad ' \tag{60}$$

A spinor of the first left ideal transforms as [4]

$$\begin{array}{l} (\psi^{11}\underline{1} + \psi^{21}\theta_1\theta_2 + \psi^{31}\theta_1 + \psi^{41}\theta_2)\,\bar\theta_1\bar\theta_2 \;\to \\ \quad(-\psi^{11}\underline{1} + \psi^{21}\bar\theta_1\theta_2 \; - \psi^{31}\bar\theta_1 + \psi^{41}\theta_2)\,\theta_1\bar\theta_2\,. \end{array} \tag{61}$$

The latter equation shows that a left handed spinor of the first ideal transforms into a right handed spinor of the third ideal.

In general, under space inversion, the matrix of the spinor basis elements

$$\xi_{\alpha i} = \begin{pmatrix} f_1 & f_2 & \bar\theta_1 f_3 & \theta_1 f_4 \\ \theta_1\theta_2 f_1 & \bar\theta_1\bar\theta_2 f_2 & \theta_2 f_3 & \bar\theta_2 f_4 \\ \theta_1 f_1 & \bar\theta_1 f_2 & f_3 & f_4 \\ \theta_2 f_1 & \bar\theta_2 f_2 & \bar\theta_1\theta_2 f_3 & \theta_1\bar\theta_2 f_4 \end{pmatrix}, \tag{62}$$

transforms into

$$\xi'_{\alpha i} = \begin{pmatrix} -f_3 & -f_4 & -\theta_1 f_1 & -\bar\theta_1 f_2 \\ \bar\theta_1\theta_2 f_3 & \theta_1\bar\theta_2 f_4 & \theta_2 f_1 & \bar\theta_2 f_2 \\ -\bar\theta_1 f_3 & -\theta_1 f_4 & -f_1 & -f_2 \\ \theta_2 f_3 & \bar\theta_2 f_4 & \theta_1\theta_2 f_1 & \bar\theta_1\bar\theta_2 f_2 \end{pmatrix}. \tag{63}$$

The matrix of components

$$\psi^{\alpha i} = \begin{pmatrix} \psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\ \psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\ \psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\ \psi^{41} & \psi^{42} & \psi^{43} & \psi^{44} \end{pmatrix}$$

transform into

$$\psi^{\alpha i} = \begin{pmatrix} -\psi^{33} & -\psi^{34} & -\psi^{31} & -\psi^{32} \\ \psi^{43} & \psi^{44} & \psi^{41} & \psi^{42} \\ -\psi^{13} & -\psi^{14} & -\psi^{11} & -\psi^{12} \\ \psi^{23} & \psi^{24} & \psi^{21} & \psi^{22} \end{pmatrix}. \tag{64}$$

By comparing (62) and (63), or by inspecting (64), we find that the spinor of the 1st ideal transforms into the spinor of the 3rd ideal, and the spinor of the 2nd ideal transforms into the spinorof the 4th ideal.

Generalized Dirac equation (Dirac-Kähler equation)

Let us now consider the Clifford algebra valued fields, $\Phi(x)$, that depend on position $x \equiv x^\mu$ in spacetime. We will assume that a field Φ satisfies the following equation [8] (see also refs. [4, 6]):

$$(i\,\gamma^\mu \partial_\mu - m)\Phi = 0, \qquad \Phi = \phi^A \gamma_A = \psi^{\tilde{A}} \xi_{\tilde{A}} = \psi^{\alpha i} \xi_{\alpha i}\,. \tag{65}$$

where γ_A is a multivector basis of $Cl(1,3)$, and $\xi_{\tilde{A}} \equiv \xi_{\alpha i}$ is a spinor basis of $Cl(1,3)$, or more precisely, of its complexified version if $\psi^{\alpha i}$ are complex-valued. Here α is the spinor index of a left minimal ideal, whereas the i runs over four left ideals of $Cl(1,3)$.

Multiplying Eq. (65) from the left by $(\xi^{\tilde{A}})^\ddagger$, where $\ddagger$ is the operation of reversion that reverses the order of vectors in a product, and using the relation

$$\langle (\xi^{\tilde{A}})^\ddagger \gamma^\mu \xi_{\tilde{B}} \rangle_S \equiv (\gamma^\mu)^{\tilde{A}}{}_{\tilde{B}}\,, \tag{66}$$

and where $\langle\ \ \rangle_S$ is the (properly normalized [9]) scalar part of an expression, we obtain the following matrix form of the equation (65):

$$\left(i\,(\gamma^\mu)^{\tilde{A}}{}_{\tilde{B}}\,\partial_\mu - m\,\delta^{\tilde{A}}{}_{\tilde{B}} \right) \psi^{\tilde{B}} = 0\,. \tag{67}$$

The 16×16 matrices can be factorized according to

$$(\gamma^\mu)^{\tilde{A}}{}_{\tilde{B}} = (\gamma^\mu)^\alpha{}_\beta\, \delta^i{}_j\,, \tag{68}$$

where $(\gamma^\mu)^\alpha{}_\beta$ are 4×4 Dirac matrices. Using the latter relation (68), we can write Eq. (67) as

$$\left(i\,(\gamma^\mu)^\alpha{}_\beta\,\partial_\mu - m\,\delta^\alpha{}_\beta \right) \psi^{\beta i} = 0\,, \tag{69}$$

or more simply,

$$(i\,\gamma^\mu \partial_\mu - m)\psi^i = 0\,. \tag{70}$$

In the last equation we have omitted the spinor index α.

The action that leads to the generalized Dirac equation (65) is

$$I = \int \mathrm{d}^4x\ \bar{\psi}^i (i\,\gamma^\mu \partial_\mu - m)\psi^j z_{ij}\,. \tag{71}$$

This is an action that describes four spinors ψ^i, belonging to the four minimal left ideals of $Cl(1,3)$. Here z_{ij} is the metric in the space of ideals. It is a part of the metric

$$(\xi_{\tilde{A}})^{\ddagger} * \xi_{\tilde{B}} = z_{\tilde{A}\tilde{B}} = z_{(\alpha i)(\beta j)} = z_{\alpha\beta} z_{ij} \tag{72}$$

of the Clifford algebra $Cl(1,3)$, represented in the basis $\xi_{\tilde{A}}$:

$$z_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad z_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{73}$$

Gauge covariant action is

$$I = \int \mathrm{d}^4x \; \bar{\psi}^i (i\,\gamma^\mu D_\mu - m)\psi^j z_{ij} \;, \qquad D_\mu \psi^i = \partial_\mu \psi^i + G_\mu{}^i{}_j \psi^j \,. \tag{74}$$

This action contains the ordinary particles and mirror particles. The first and the second columns of the matrix $\psi^{\alpha i}$, written explicitly in eq. (30) describe the ordinary particles, whereas the third and the forth column in (30) describe *mirror particles*.

The SU(2) gauge group acting within the 1st and 2nd ideal can be interpreted as the weak interaction gauge group for ordinary particles. The SU(2) gauge group acting within the 3rd and 4th ideal can be interpreted as the weak interaction gauge group for mirror particles. The corresponding two kinds of weak interaction gauge fields that can be transformed into each other by space inversion are contained in $G_\mu{}^i{}_j$, which is a generalized gauge field occurring in the covariant action (74).

Mirror particles were first proposed by Lee and Yang [10]. Subsequently, the idea of mirror particles has been pursued by Kobzarev et al. [11], and in Refs. [12]– [17]. The possibility that mirror particles are responsible for dark matter has been explored in many works, e.g., in [18]– [25]. A demonstration that mirror particles can be explained in terms of algebraic spinors (elements of Clifford algebras) was presented in Ref. [4].

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On reference frames and the definition of space in a general spacetime

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Abstract: *First, we review local concepts defined previously. A (local) reference frame* F *can be defined as an equivalence class of admissible spacetime charts (coordinate systems) having a common domain* U *and exchanging by a spatial coordinate change. The associated (local) physical space is made of the world lines having constant space coordinates in any chart of the class. Second, we introduce new, global concepts. The data of a non-vanishing global vector field* v *defines a global "reference fluid". The associated global physical space is made of the maximal integral curves of that vector field. Assume that, in any of the charts which make some reference frame* F*: (i) any of those integral curves* l *has constant space coordinates* x^j*, and (ii) the mapping* $l \mapsto (x^j)$ *is one-to-one. In that case, the local space can be identified with a part (an open subset) of the global space.*

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Introduction

A reference frame, in a broad physical sense, is a three-dimensional network of observers equipped with clocks and meters. To any reference frame one should be able to associate some three-dimensional *space,* in which the observers of the network are by definition at rest (even though their mutual distances may depend on time). Clearly, both notions are fundamental ones for physics. In Newtonian physics, the consideration is in general (though not always [1, 2]) restricted to reference frames that are *rigid* with respect to the invariant Euclidean space metric. The same restriction is used in special relativity: there, one considers mainly the *inertial frames,* each of which is rigid with respect to the *spatial metric* in the considered reference frame.

In the relativistic theories of gravitation, the main object is the spacetime metric, which is a field, i.e. it depends on the spacetime position. Hence, rigid reference frames are not relevant any more. The relevant notion is that of a *reference fluid.* A three-dimensional network of observers is defined by *a time-like vector field* v *on spacetime* [3–6]: v is the unit tangent vector field to the world lines of the observers belonging to the network. However, in the general-relativistic literature, it is often implicit that a reference frame can be defined from the data of a *coordinate system* (or *chart*); see e.g. Landau & Lifshitz [7] and Møller [8]. The link with the definition by a 4-velocity vector field v was done by Cattaneo [3]. Namely, any admissible chart on the spacetime, $\chi : X \mapsto (x^\mu)$ $(\mu = 0, ..., 3)$, defines a unique reference fluid, given by its four-velocity field v: the components of v in the chart χ are

$$v^0 \equiv \frac{1}{\sqrt{g_{00}}}, \qquad v^j = 0 \quad (j = 1, 2, 3). \tag{1}$$

The vector (1) is invariant under the "internal changes"

$$x'^0 = \phi((x^\mu)), \quad x'^k = \phi^k((x^j)) \quad (j, k = 1, 2, 3). \tag{2}$$

We note, however, that this is valid only within the domain of definition of the chart χ — an open subset U of the whole spacetime manifold V.

The notion of the *space associated* with a reference fluid/network was missing in the general-relativistic literature. However, it is apparent in experimental or observational papers that one cannot dispense with the notion of a spatial position (of any part of the experimental apparatus and the observed system). In the absence

of a definite concept of space, such a position is defined by a set of spatial coordinates. This is not satisfactory, because many different coordinate systems can be defined, between which the choice seems arbitrary. One needs to have a theoretical framework that give a precise meaning to the concept of the space associated with a reference fluid/network. Only a concept of "spatial tensor" had been defined, to our knowledge. Namely, a spatial tensor at $X \in \mathrm{V}$ was defined as a *spacetime tensor* which equals its projection onto the hyperplane $\mathrm{H}_X \equiv v(X)^\perp$ [4, 9]. This is not a very straightforward definition. In addition, a number of time derivatives along a trajectory can then be introduced [9]. It is difficult to choose among them.

In this conference paper, we will first recall the results obtained previously [10, 11] regarding the definition of a local reference frame and the associated local space. Then we will announce results of a current work, that aims at defining global notions and at relating them to the formerly introduced local notions.

A local definition of a reference frame and the associated space

Defining a "reference fluid" through its 4-velocity field is correct but unpractical. On the other hand, fixing a "reference system" by the data of a chart [7, 8] is practical, but one may ask: what is physical here? Is there an associated space? What if we change the chart?

Space associated with a reference fluid: a sketch

The three-dimensional space manifold N associated with a reference fluid (network of observers) can be introduced as *the set of the world lines of the observers of the network* [10]. Thus an *element* (point) of N is a *line* of the spacetime manifold V. Spatial tensor fields are then defined simply as tensor fields on the spatial manifold N [10]. At the time of that definition [10], the network, hence also N, was thus defined "physically", and it was not proved that N is indeed a differentiable manifold. Nevertheless, it was noted that the spatial metric defined in Refs. [7] and [8] endows this manifold N with a time-dependent Riemannian metric, thus with a one-parameter family of metrics. Then, just one time derivative along a trajectory appears naturally [10], precisely because we have now just a time- dependent spatial metric tensor instead of a general spacetime metric. This allowed us to unambiguously define *Newton's second law* in a general spacetime.

A local definition of a reference frame

One may define a reference frame as being an *equivalence class of charts* which are all defined on a given open subset U of the spacetime V and are related two-by-two by a *purely spatial* coordinate change:

$$x'^0 = x^0, \quad x'^k = \phi^k((x^j)). \tag{3}$$

This does define an equivalence relation [11]. Thus a reference frame F, i.e. an equivalence class for this relation, can indeed be given by *the data of one chart* $\chi : X \mapsto (x^\mu)$ *with its domain of definition* U (an open subset of the spacetime manifold V). Namely, F is the equivalence class of (χ, U). I.e., F is the set of the charts χ' which are defined on U, and which are such that the transition map $f \equiv \chi' \circ \chi^{-1} \equiv (\phi^\mu)$ corresponds with a purely spatial coordinate change (3).

The associated space

The former definition has physical meaning: the data of a reference frame F determines the world lines (each of which is included in the common chart domain U):

$$x^j = \text{Constant } (j = 1, 2, 3), \ x^0 \text{ variable.} \tag{4}$$

The set of these world lines, as $\mathbf{x} \equiv (x^j)$ varies, is indeed a three-dimensional network. If the charts obey the admissibility condition $g_{00} > 0$, these are time-like world lines. The corresponding 4-velocity field v or rather v_F is then given by (1). The world lines (4) as well as the field v_F are invariant under the "internal changes" (2). Hence, they are a fortiori invariant under the purely spatial coordinate changes (3). The space $\mathrm{M} = \mathrm{M}_\mathrm{F}$ (in a further step to be equipped with a structure of differentiable manifold) is mathematically defined as the

set of the world lines (4).

In full detail: let $P_S : \mathbb{R}^4 \to \mathbb{R}^3, \quad \mathbf{X} \equiv (x^\mu) \mapsto \mathbf{x} \equiv (x^j)$, be the spatial projection. A world line l is an element of the set $\mathrm{M_F}$ iff there is a chart $\chi \in \mathrm{F}$ and a triplet $\mathbf{x} \equiv (x^j) \in \mathbb{R}^3$, such that l is the set, assumed non-empty, of *all* points X in the domain U, whose spatial coordinates are $\mathbf{x}$:

$$l = \{\, X \in \mathrm{U};\ P_S(\chi(X)) = \mathbf{x} \,\} \quad \text{and } l \neq \emptyset. \tag{5}$$

Note that the lines (5), hence also their set $\mathrm{M_F}$, remain invariant, not only under the purely spatial coordinate changes (3), but under any change (2). The coordinate changes (2) leave the 4-velocity field v_F invariant, but in general they change the reference frame, say from F to F′, since they generally change the time coordinate. In such a case, we have thus $\mathrm{M_F} = \mathrm{M_{F'}}$.

$\mathrm{M_F}$ is a differentiable manifold: sketch of the proof

Consider a chart $\chi \in \mathrm{F}$. With any world line $l \in \mathrm{M_F}$, let us associate the triplet $\mathbf{x} \equiv (x^j)$ made with the *constant* spatial coordinates of the points $X \in l$. We thus define a mapping

$$\widetilde{\chi} : \mathrm{M_F} \to \mathbb{R}^3, \quad l \mapsto \mathbf{x} \text{ such that } \forall X \in l,\ \chi^j(X) = x^j \ (j = 1, 2, 3). \tag{6}$$

Through Eq. (5), the world line $l \in \mathrm{M_F}$ is determined uniquely by the data $\mathbf{x}$. I.e., the mapping $\widetilde{\chi}$ is one-to-one. Consider the set $\mathcal{T}$ of the subsets $\Omega \subset \mathrm{M_F}$ such that

$$\forall \chi \in \mathrm{F}, \quad \widetilde{\chi}(\Omega) \text{ is an open set in } \mathbb{R}^3. \tag{7}$$

One shows that $\mathcal{T}$ is a topology on $\mathrm{M_F}$. Then one shows that the set of the mappings $\widetilde{\chi}$ defines a structure of differentiable manifold on that topological space $\mathrm{M_F}$: *The spatial part of any chart $\chi \in \mathrm{F}$ defines a chart $\widetilde{\chi}$ on* $\mathrm{M_F}$ [11]. In particular, the compatibility of any two charts $\widetilde{\chi}$ and $\widetilde{\chi}'$ on $\mathrm{M_F}$ stems from the fact that any two charts χ, χ' that belong to one reference frame F have a common domain U: since any world line $l \in \mathrm{M_F}$ is included in U, one shows easily that $\widetilde{\chi}' \circ \widetilde{\chi}^{-1} = (\phi^k)$, the spatial part of the transition map $\chi' \circ \chi^{-1}$.

Applications of this result

A Hamiltonian operator of relativistic QM depends *precisely* [12] on the reference frame F as defined in Subsect. . The Hilbert space $\mathcal{H}$ of quantum-mechanical states is the set of the square-integrable functions defined on the associated *space manifold* $\mathrm{M_F}$ [13]. Prior to this definition, $\mathcal{H}$ depended on the particular spatial coordinate system. This does not seem acceptable.

The full algebra of spatial tensors can then be defined in a simple way: a spatial tensor field is simply a tensor field on the space manifold $\mathrm{M_F}$ associated with a reference frame F. A simple example is the *3-velocity* of a particle (or a volume element) in a reference frame: this is a spatial vector, i.e., the current 3-velocity at an event $X \in \mathrm{U}$ is an element of the tangent space at $l(X) \in \mathrm{M_F}$. {$l(X)$ is the unique line $l \in \mathrm{M_F}$, such that $X \in l$ [11].} As another example, the *rotation rate of a spatial triad* is an antisymmetric spatial tensor field of the (0 2) type [14].

Questions left open by that result

These definitions of a *reference frame* and the associated space manifold apply to a domain U of the spacetime V, such that at least one regular chart can be defined over the whole of U. Thus these are *local* definitions, since in general the whole spacetime manifold V cannot be covered by a single chart. Whence the questions:

Can the definition of a *reference fluid* by the data of a *global* four-velocity field v lead to a global notion of space? If yes, what is the link with the former local notions?

The global space manifold N_v associated with a non-vanishing vector field v

Given a global vector field v on the spacetime V, and given an event $X \in \mathrm{V}$, let C_X be the solution of

$$\frac{dC}{ds} = v(C(s)), \qquad C(0) = X \tag{8}$$

that is defined on the *largest possible* open interval I_X containing 0 [15]. Call the *range* $l_X \equiv C_X(\mathrm{I}_X) \subset \mathrm{V}$ the "maximal integral curve at X". If $X' \in l_X$, then it is easy to show that $l_{X'} = l_X$.

We define the *global space* N_v *associated with the vector field* v as the set of the maximal integral curves of v:

$$\mathrm{N}_v \equiv \{l_X;\ X \in \mathrm{V}\}. \tag{9}$$

Local existence of adapted charts

A chart χ with domain $\mathrm{U} \subset \mathrm{V}$ is said v*–adapted* iff the spatial coordinates remain constant on any integral curve l of v — more precisely, remain constant on $l \cap \mathrm{U}$:

$$\forall l \in \mathrm{N}_v,\ \exists \mathbf{x} \equiv (x^j) \in \mathbb{R}^3 :$$

$$\forall X \in l \cap \mathrm{U}, \quad P_S(\chi(X)) = \mathbf{x}. \tag{10}$$

For any v–adapted chart χ, the mapping

$$\bar{\chi} : l \mapsto \mathbf{x} \text{ such that (10) is verified} \tag{11}$$

is well defined on

$$\mathrm{D}_\mathrm{U} \equiv \{l \in \mathrm{N}_v;\ l \cap \mathrm{U} \neq \emptyset\}. \tag{12}$$

Call the v–adapted chart χ *nice* if the mapping $\bar{\chi}$ is one-to-one. On the other hand, call a non-vanishing [1] global vector field v *normal* if its flow has the following property that, we can indicate convincingly, is true unless v is "pathological": *Any point* $X \in \mathrm{V}$ *has an open neighborhood* U *such that: (i) for any maximal integral curve* l *of* v*, the intersection* $l \cap \mathrm{U}$ *is a connected set, and (ii) there is a chart* χ *with domain* U*, such that the corresponding natural basis* (∂_μ) *verifies* $v = \partial_0$ *in* U. It is easy to prove the following:

Theorem 1. *Let the global non-vanishing vector field* v *on* V *be normal. Then, for any point* $X \in \mathrm{V}$*, there exists a nice* v*–adapted chart* χ *whose domain is an open neighborhood of* X*.*

Manifold structure of the global set N_v

Consider the set $\mathcal{F}_v$ made of all nice v–adapted charts on the spacetime manifold V, and consider the set $\mathcal{A}$ made of the mappings $\bar{\chi}$, where $\chi \in \mathcal{F}_v$, Eq. (11). A such mapping $\bar{\chi}$ is defined on the set D_U — a subset of the three-dimensional "space" N_v, Eq. (12). (Here U is the domain of the v–adapted chart $\chi \in \mathcal{F}_v$.) When Theorem 1 above applies, we can go further:

First, in exactly the same way as that used [11] to prove that the set $\mathcal{T}$ (7) is a topology on the "local" space M_F, we can show that the set $\mathcal{T}'$ of the subsets $\Omega \subset \mathrm{N}_v$ such that

$$\forall \chi \in \mathcal{F}_v, \quad \bar{\chi}(\Omega) \text{ is an open set in } \mathbb{R}^3, \tag{13}$$

[1] Note that a time-like vector field is non-vanishing. However, we don't need that v be time-like.

is a topology on the global space N_v. [We define $\bar{\chi}(\Omega) \equiv \bar{\chi}(\Omega \cap \mathrm{D_U})$.]

Second, we can show that $\mathcal{A}$ is an atlas on that topological space, thus defining a structure of differentiable manifold on the global set N_v. In order to show this, the main thing to prove is the compatibility of any two charts $\bar{\chi}$, $\bar{\chi}'$ on N_v, associated with two nice v-adapted charts $\chi, \chi' \in \mathcal{F}_v$.

In the case of the space manifold $\mathrm{M_F}$ associated with a local reference frame F, the compatibility of two associated charts $\widetilde{\chi}$ and $\widetilde{\chi}'$ on $\mathrm{M_F}$ was rather easy to prove, see the end of Sect. . In contrast, two v-adapted charts χ and χ' have in general different domains U and U′ and we may have

$$\mathrm{U} \cap \mathrm{U}' = \emptyset, \quad l \cap \mathrm{U} \neq \emptyset, \quad l \cap \mathrm{U}' \neq \emptyset. \tag{14}$$

I.e., the domains of the charts χ and χ' do not overlap, but the domains of the mappings $\bar{\chi}$ and $\bar{\chi}'$ do. The solution of this difficulty can be sketched as follows. Consider $\mathbf{x} \in \mathrm{Dom}(\bar{\chi}' \circ \bar{\chi}^{-1}) = \bar{\chi}(\mathrm{D_U} \cap \mathrm{D_{U'}})$. Since $\mathbf{x} \in \bar{\chi}(\mathrm{D_U})$, $\exists l \in \mathrm{N}_v$ and $\exists X \in l \cap \mathrm{U}$: $\mathbf{x} = \bar{\chi}(l) = P_S(\chi(X))$. Let $\chi(X) = (t, \mathbf{x})$. We use the flow of the vector field v to associate smoothly with any point Y in some neighborhood $\mathrm{W} \subset \mathrm{U}$ of X, a point $g(Y) \in \mathrm{U}'$. Then we may write for $\mathbf{y}$ in a neighborhood of $\mathbf{x}$:

$$\left(\bar{\chi}' \circ \bar{\chi}^{-1}\right)(\mathbf{y}) = P_S(\chi'(g(\chi^{-1}(t, \mathbf{y})))), \tag{15}$$

showing the smoothness of $\bar{\chi}' \circ \bar{\chi}^{-1}$. Using this, we show that the set $\mathcal{A}$ of the mappings $\bar{\chi}$ is an atlas on N_v, making it a differentiable manifold.

The local manifold $\mathrm{M_F}$ is a submanifold of N_v

Let v be a normal non-vanishing vector field on V, and let F be a reference frame *made of nice v–adapted charts,* all defined on the same open set $\mathrm{U} \subset \mathrm{V}$.

Let $l \in \mathrm{M_F}$, thus there is some chart $\chi \in \mathrm{F}$ and some $\mathbf{x} \in \mathbb{R}^3$ such that $l = \{\, X \in \mathrm{U};\ P_S(\chi(X)) = \mathbf{x} \,\}$. Then, for any $X \in l$, the curve l_X is the same maximal integral curve $l' \in \mathrm{N}_v$, and we have $l = l' \cap \mathrm{U}$. We have moreover $l' = \bar{\chi}^{-1}(\mathbf{x}) = \bar{\chi}^{-1}(\widetilde{\chi}(l))$. Hence, the mapping $I : \mathrm{M_F} \to \mathrm{N}_v,\ l \mapsto l'$ is just $I = \bar{\chi}^{-1} \circ \widetilde{\chi}$. This one-to-one mapping of $\mathrm{Dom}(\widetilde{\chi}) = \mathrm{M_F}$ onto $\mathrm{Dom}(\bar{\chi}) = \mathrm{D_U}$ is a diffeomorphism, hence it is an immersion of $\mathrm{M_F}$ into N_v. Thus $\mathrm{M_F}$ is made of the intersections with the local domain U of the world lines belonging to N_v, and *we may identify the local space* $\mathrm{M_F}$ *with the submanifold* $I(\mathrm{M_F}) = \mathrm{D_U}$ *of the global space* N_v. Now the manifold structure of N_v entails that, for any nice v-adapted chart $\chi \in \mathcal{F}_v$, the associated mapping $\bar{\chi}$ with domain $\mathrm{D_U}$ is a chart on the topological space $(\mathrm{N}_v, \mathcal{T}')$. In turn, this fact involves the statement that $\mathrm{D_U}$ is more specifically an *open subset* of N_v.

Conclusion

A reference frame can be defined as an equivalence class of spacetime charts χ which have a common domain U and which exchange two-by-two by a purely spatial coordinate change [11]. In addition to being mathematically correct, this definition is practical, because it gives a methodology to use coordinate systems in a consistent and physically meaningful way: the data of one spacetime coordinate system (x^μ) defines (in its domain of definition U) the 4-velocity field of a network of observers, Eq. (1). The coordinate systems that exchange with (x^μ) by a purely spatial coordinate change (3) belong to the same reference frame and indeed the associated 4-velocity field (1) is the same. Using a general coordinate change instead, allows us to go to any other possible reference frame.

A precise notion of the physical *space* associated with a given reference network did not exist before for a general spacetime, to our knowledge. We defined two distinct concepts: a local one and a global one, which however are intimately related together. In either case, the space is the set of the world lines that belong to the given (local) reference frame, respectively to the given (global) reference fluid:

i) Consider a (local) reference frame in the specific sense meant here, i.e. a set F of charts, all defined on the same subdomain U of the spacetime, and exchanging by a change of the form (3). This allows one to define

a "local space" $\mathrm{M_F}$: this is the set of the world lines (4) [more precisely the set of the world lines (5)] [11]. Each of these world lines is included in the common domain U of all charts $\chi \in \mathrm{F}$.

ii) The data of a (global) reference fluid, i.e. a global non-vanishing 4-vector field v, allows one to define a "global space" N_v: this is the set of the maximal integral curves of v.

Both of the local space $\mathrm{M_F}$ and the global space N_v can be endowed with a structure of differentiable manifold (when Theorem 1 applies, for the global space). The manifold structure gives a firm status to the space attached to a reference network and allows us to define spatial tensors naturally, as tensor fields on the space manifold. It has also a practical aspect: Locally, the position of a point in the space can be specified by different sets of spatial coordinates, which exchange smoothly: $x'^k = \phi^k((x^j)) \quad (j,k = 1,2,3)$, and we may use standard differential calculus for mappings defined on that space, by choosing any such coordinates. This applies to both the local space $\mathrm{M_F}$ and the global space N_v.

There is a close link between the local space $\mathrm{M_F}$ and the global space N_v, provided the three-dimensional network of observers is indeed the same in the two cases — i.e., provided that, in any of the charts which make the reference frame F: (i) any of the integral curves $l \in \mathrm{N}_v$ has constant space coordinates x^j, and (ii) the mapping $l \mapsto (x^j)$ is one-to-one. If that is true, one may associate with each world line $l \in \mathrm{M_F}$ the world line $l' \in \mathrm{N}_v$, of which l is just the intersection with the domain U. Thus the local space can be identified with an open subset of the global space.

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Emerging of massive gauge particles in inhomogeneous local gauge transformations

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Abstract: *A generalized theory of gauge transformations is presented on the basis of the covariant Hamiltonian formalism of field theory, for which the covariant canonical field equations are equivalent to the Euler-Lagrange field equations. Similar to the canonical transformation theory of point dynamics, the canonical transformation rules for fields are derived from generating functions. Thus—in contrast to the usual Lagrangian description—the covariant canonical transformation formalism automatically ensures the mappings to preserve the action principle, and hence to be* physical. *On that basis, we work out the theory of* inhomogeneous *local gauge transformations that generalizes the conventional local SU(N) gauge transformation theory. It is shown that massive gauge bosons naturally emerge in this description, which thus could supersede the Higgs mechanism.*

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"Die Fruchtbarkeit des neuen Gesichtspunktes der Eichinvarianz hätte sich vor allem am Problem der Materie zu zeigen." (Weyl 1919 [1])
"The fruitfulness of the new viewpoint of gauge invariance would have to show up in particular on the problem of matter."

Introduction

The principle of *local gauge invariance* has been proven to be an eminently fruitful device for deducing all elementary particle interactions within the standard model. On the other hand, the gauge principle is justified only as far as it "works": a deeper rationale underlying the gauge principle apparently does not exist. In this respect, the gauge principle corresponds to other basic principles of physics, such as Fermat's "principle of least time", the "principle of least action" as well as its quantum generalization leading to Feynman's path integral formalism. The failure of the conventional gauge principle to explain the existence of massive gauge bosons has led to *supplementing* it with the Higgs-Kibble mechanism [2, 3].

An alternative strategy to resolve the mass problem would be to directly generalize the conventional gauge principle in a natural way. One way to achieve this was to require the system's covariant Hamiltonian to be form-invariant not only under unitary transformations of the fields in iso-space, but also under variations of the space-time metric. This idea of a generalization of the conventional gauge principle has been successfully worked out and was published recently [4]. In this description, the gauge field causes a non-vanishing curvature tensor, and this curvature tensor appears in the field equations as a mass factor.

With the actual paper, a second natural generalization of the conventional gauge transformation formalism will be presented that extends the conventional SU(N) gauge theory to include *inhomogeneous* linear mappings of the fields. As it turns out, the local gauge-invariance of the system's Lagrangian then requires the existence of massive gauge fields, with the mass playing the role of a second coupling constant. We thereby tackle the long-standing inconsistency of the *conventional* gauge principle that requires gauge bosons to be massless in order for any theory to be locally gauge-invariant. This will be achieved *without* postulating a particular potential function ("Mexican hat") and without requiring a "symmetry breaking" phenomenon.

Conventional gauge theories are commonly derived on the basis of Lagrangians of relativistic field theory (cf, for instance, [5–7]). Although perfectly valid, the Lagrangian formulation of gauge transformation theory is *not*

the optimum choice. The reason is that in order for a Lagrangian transformation theory to be physical, hence to maintain the action principle, it must be supplemented by additional structure, referred to as the *minimum coupling rule*.

In contrast, the formulation of gauge theories in terms of covariant Hamiltonians (cf, for instance, Kanatchikov [8])—each of them being equivalent to a corresponding Lagrangian—may exploit the framework of the *canonical transformation formalism*. With the transformation rules for all fields and their canonical conjugates being derived from *generating functions*, we restrict ourselves from the outset to exactly the *subset* of transformations that preserve the action principle, hence ensure the actual gauge transformation to be *physical*. No additional structure needs to be incorporated for setting up an amended Hamiltonian that is *locally* gauge-invariant on the basis of a given *globally* gauge-invariant Hamiltonian. The *covariant derivative*—defined by the *minimum coupling rule*—automatically arises as the respective *canonical momentum*. Furthermore, it is no longer required to postulate the field tensor to be skew-symmetric in its space-time indices as this feature directly emerges from the canonical transformation formalism.

Prior to working out the inhomogeneous local gauge transformation theory in the covariant Hamiltonian formalism—the latter dating back to DeDonder [9] and Weyl [10]—a concise review of the concept of covariant Hamiltonians in local coordinate representation is outlined in . Thereafter, the canonical transformation formalism in the realm of field theory is sketched briefly in . In these sections, we restrict our presentation to exactly those topics of the canonical formalism that are essential for working out the inhomogeneous gauge transformation theory, which will finally be covered in .

The requirement of *inhomogeneous* local gauge invariance naturally generalizes the conventional SU(N) gauge principle (cf, for instance, [11]), where the form-invariance of the covariant Hamiltonian density is demanded under *homogeneous* unitary mappings of the fields in iso-space. In the first step, a generating function of type $\boldsymbol{F}_2$ is set up that merely describes the demanded transformation of the fields in iso-space. As usual, this transformation forces us to introduce gauge fields that render an appropriately amended Hamiltonian locally gauge-invariant if the gauge fields follow a particular transformation law. In our case of an inhomogeneous mapping, we are forced to introduce *two* independent sets of gauge fields, each of them requiring its own transformation law.

In the second step, an *amended* generating function $\boldsymbol{F}_2$ is constructed in a way to define these transformation laws for the two sets of gauge fields in addition to the rules for the base fields. As the characteristic feature of the canonical transformation formalism, this amended generating function also provides the transformation law for the conjugates of the gauge fields and for the Hamiltonian. This way, we derive the Hamiltonian that is form-invariant under both the inhomogeneous mappings of the base fields as well as under the required mappings of the two sets of gauge fields.

In a third step, it must be ensured that the canonical field equations emerging from the gauge-invariant Hamiltonian are consistent with the canonical transformation rules. As usual in gauge theories, the Hamiltonian must be further amended by terms that describe the free-field dynamics of the gauge fields while maintaining the overall form-invariance of the final Hamiltonian. Amazingly, this also works for our inhomogeneous gauge transformation theory and *uniquely* determines the final gauge-invariant Hamiltonian $\mathcal{H}_3$.

The Hamiltonian $\mathcal{H}_3$ is then Legendre-transformed to yield the equivalent gauge-invariant Lagrangian density $\mathcal{L}_3$. The latter can then serve as the starting point to set up the Feynman diagrams for the various mutual interactions of base and gauge fields. As an example, the locally gauge-invariant Lagrangian that emerges from a base system of an N-tuple of massless spin-0 fields is presented.

Covariant Hamiltonian density

In field theory, the Hamiltonian density is usually defined by performing an *incomplete* Legendre transformation of a Lagrangian density $\mathcal{L}$ that only maps the time derivative $\partial_t\phi$ of a field $\phi(t,x,y,z)$ into a corresponding canonical momentum variable, π_t. Taking then the spatial integrals results in a description that corresponds to that of non-relativistic Hamiltonian point dynamics. Yet, in analogy to relativistic point dynamics [12], a covariant Hamiltonian description of field theory must treat space and time variables on equal footing. If $\mathcal{L}$ is a Lorentz scalar, this property is passed to the *covariant DeDonder-Weyl Hamiltonian density* $\mathcal{H}$ that emerges from a *complete* Legendre transformation of $\mathcal{L}$. Moreover, this description enables us to devise a consistent theory of canonical transformations in the realm of classical field theory.

Covariant canonical field equations

The transition from particle dynamics to the dynamics of a *continuous* system is based on the assumption that a *continuum limit* exists for the given physical problem (cf, for instance, [13]). This limit is defined by letting the number of particles involved in the system increase over all bounds while letting their masses and distances go to zero. In this limit, the information on the location of individual particles is replaced by the *value* of a smooth function $\phi(x)$ that is given at a spatial location x^1, x^2, x^3 at time $t \equiv x^0$. In this notation, the index μ runs from 0 to 3, hence distinguishes the four independent variables of space-time $x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv (t, x, y, z)$, and $x_\mu \equiv (x_0, x_1, x_2, x_3) \equiv (t, -x, -y, -z)$. We furthermore assume that the given physical problem can be described in terms of a set of $I = 1, \ldots, N$—possibly interacting—scalar fields $\phi_I(x)$, with the index "I" enumerating the individual fields. A transformation of the fields in iso-space is not associated with any non-trivial metric. We, therefore, do not use superscripts for these indices as there is not distinction between covariant and contravariant components. In contrast, Greek indices are used for those components that *are* associated with a metric—such as the derivatives with respect to a space-time variable, x^μ. Throughout the article, the summation convention is used. Whenever no confusion can arise, we omit the indices in the argument list of functions in order to avoid the number of indices to proliferate.

The Lagrangian description of the dynamics of a continuous system is based on the Lagrangian density function $\mathcal{L}$ that is supposed to carry the complete information on the given physical system. In a first-order field theory, the Lagrangian density $\mathcal{L}$ is defined to depend on the ϕ_I, possibly on the vector of independent variables x^μ, and on the four first derivatives of the fields ϕ_I with respect to the independent variables, i.e., on the 1-forms (covectors)

$$\partial_\mu \phi_I \equiv (\partial_t \phi_I, \partial_x \phi_I, \partial_y \phi_I, \partial_z \phi_I).$$

The Euler-Lagrange field equations are then obtained as the zero of the variation δS of the action integral

$$S = \int \mathcal{L}(\phi_I, \partial_\mu \phi_I, x)\, d^4x \tag{1}$$

as

$$\frac{\partial}{\partial x^\alpha} \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_I)} - \frac{\partial \mathcal{L}}{\partial \phi_I} = 0. \tag{2}$$

To derive the equivalent *covariant* Hamiltonian description of continuum dynamics, we first define for each field $\phi_I(x)$ a 4-vector of conjugate momentum fields $\pi_I^\mu(x)$. Its components are given by

$$\pi_I^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_I)} \equiv \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi_I}{\partial x^\mu}\right)}. \tag{3}$$

The 4-vector π_I^μ is thus induced by the Lagrangian $\mathcal{L}$ as the *dual counterpart* of the 1-form $\partial_\mu \phi_I$. For the entire set of N scalar fields $\phi_I(x)$, this establishes a set of N conjugate 4-vector fields. With this definition of the 4-vectors of canonical momenta $\pi_I(x)$, we can now define the Hamiltonian density $\mathcal{H}(\phi_I, \pi_I, x)$ as the covariant Legendre transform of the Lagrangian density $\mathcal{L}(\phi_I, \partial_\mu \phi_I, x)$

$$\mathcal{H}(\phi_I, \pi_I, x) = \pi_J^\alpha \frac{\partial \phi_J}{\partial x^\alpha} - \mathcal{L}(\phi_I, \partial_\mu \phi_I, x). \tag{4}$$

In order for the Hamiltonian $\mathcal{H}$ to be valid, we must require the Legendre transformation to be *regular*, which means that for each index "I" the Hesse matrices $(\partial^2 \mathcal{L}/\partial(\partial^\mu \phi_I)\, \partial(\partial_\nu \phi_I))$ are non-singular. This ensures that by means of the Legendre transformation, the Hamiltonian $\mathcal{H}$ takes over the complete information on the given dynamical system from the Lagrangian $\mathcal{L}$. The definition of $\mathcal{H}$ by Eq. (4) is referred to in literature as the "DeDonder-Weyl" Hamiltonian density.

Obviously, the dependencies of $\mathcal{H}$ and $\mathcal{L}$ on the ϕ_I and the x^μ only differ by a sign,

$$\left.\frac{\partial \mathcal{H}}{\partial x^\mu}\right|_{\text{expl}} = -\left.\frac{\partial \mathcal{L}}{\partial x^\mu}\right|_{\text{expl}}, \qquad \frac{\partial \mathcal{H}}{\partial \phi_I} = -\frac{\partial \mathcal{L}}{\partial \phi_I} = -\frac{\partial}{\partial x^\alpha}\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_I)} = -\frac{\partial \pi_I^\alpha}{\partial x^\alpha}.$$

These variables thus do not take part in the Legendre transformation of Eqs. (3), (4). Thus, with respect to this transformation, the Lagrangian density $\mathcal{L}$ represents a function of the $\partial_\mu \phi_I$ only and does *not depend* on

the canonical momenta π_I^μ, whereas the Hamiltonian density $\mathcal{H}$ is to be considered as a function of the π_I^μ only and does not depend on the derivatives $\partial_\mu \phi_I$ of the fields. In order to derive the second canonical field equation, we calculate from Eq. (4) the partial derivative of $\mathcal{H}$ with respect to π_I^μ,

$$\frac{\partial \mathcal{H}}{\partial \pi_I^\mu} = \delta_{IJ}\, \delta_\mu^\alpha \frac{\partial \phi_J}{\partial x^\alpha} = \frac{\partial \phi_I}{\partial x^\mu} \qquad \Longleftrightarrow \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_I)} = \pi_J^\alpha \delta_{IJ}\, \delta_\alpha^\mu = \pi_I^\mu.$$

The complete set of covariant canonical field equations is thus given by

$$\frac{\partial \mathcal{H}}{\partial \pi_I^\mu} = \frac{\partial \phi_I}{\partial x^\mu}, \qquad \frac{\partial \mathcal{H}}{\partial \phi_I} = -\frac{\partial \pi_I^\alpha}{\partial x^\alpha}. \tag{5}$$

This pair of first-order partial differential equations is equivalent to the set of second-order differential equations of Eq. (2). We observe that in this formulation of the canonical field equations, all coordinates of space-time appear symmetrically—similar to the Lagrangian formulation of Eq. (2). Provided that the Lagrangian density $\mathcal{L}$ is a Lorentz scalar, the dynamics of the fields is invariant with respect to Lorentz transformations. The covariant Legendre transformation (4) passes this property to the Hamiltonian density $\mathcal{H}$. It thus ensures *a priori* the relativistic invariance of the fields that emerge as integrals of the canonical field equations if $\mathcal{L}$—and hence $\mathcal{H}$—represents a Lorentz scalar.

Canonical transformations in covariant Hamiltonian field theory

The covariant Legendre transformation (4) allows us to derive a canonical transformation theory in a way similar to that of point dynamics. The main difference is that now the generating function of the canonical transformation is represented by a *vector* rather than by a scalar function. The main benefit of this formalism is that we are not dealing with arbitrary transformations. Instead, we restrict ourselves *right from the beginning* to those transformations that preserve the form of the action functional. This ensures all eligible transformations to be *physical*. Furthermore, with a generating function, we not only define the transformations of the fields but also pinpoint simultaneously the corresponding transformation law of the canonical momentum fields.

Generating functions of type $\mathbf{F}_1(\phi, \mathbf{\Phi}, x)$

Similar to the canonical formalism of point mechanics, we call a transformation of the fields $(\phi, \pi) \mapsto (\mathbf{\Phi}, \mathbf{\Pi})$ *canonical* if the form of the variational principle that is based on the action functional (1) is maintained,

$$\delta \int_R \left(\pi_I^\alpha \frac{\partial \phi_I}{\partial x^\alpha} - \mathcal{H}(\phi, \pi, x) \right) d^4x \stackrel{!}{=} \delta \int_R \left(\Pi_I^\alpha \frac{\partial \Phi_I}{\partial x^\alpha} - \mathcal{H}'(\mathbf{\Phi}, \mathbf{\Pi}, x) \right) d^4x. \tag{6}$$

Equation (6) tells us that the *integrands* may differ by the divergence of a vector field F_1^μ, whose variation vanishes on the boundary ∂R of the integration region R within space-time

$$\delta \int_R \frac{\partial F_1^\alpha}{\partial x^\alpha} d^4x = \delta \oint_{\partial R} F_1^\alpha dS_\alpha \stackrel{!}{=} 0.$$

The immediate consequence of the form invariance of the variational principle is the form invariance of the covariant canonical field equations (5)

$$\frac{\partial \mathcal{H}'}{\partial \Pi_I^\mu} = \frac{\partial \Phi_I}{\partial x^\mu}, \qquad \frac{\partial \mathcal{H}'}{\partial \Phi_I} = -\frac{\partial \Pi_I^\alpha}{\partial x^\alpha}.$$

For the integrands of Eq. (6)—hence for the Lagrangian densities $\mathcal{L}$ and $\mathcal{L}'$—we thus obtain the condition

$$\begin{aligned} \mathcal{L} &= \mathcal{L}' + \frac{\partial F_1^\alpha}{\partial x^\alpha} \\ \pi_I^\alpha \frac{\partial \phi_I}{\partial x^\alpha} - \mathcal{H}(\phi, \pi, x) &= \Pi_I^\alpha \frac{\partial \Phi_I}{\partial x^\alpha} - \mathcal{H}'(\mathbf{\Phi}, \mathbf{\Pi}, x) + \frac{\partial F_1^\alpha}{\partial x^\alpha}. \end{aligned} \tag{7}$$

With the definition $F_1^\mu \equiv F_1^\mu(\phi, \mathbf{\Phi}, x)$, we restrict ourselves to a function of exactly those arguments that now enter into transformation rules for the transition from the original to the new fields. The divergence of F_1^μ writes, explicitly,

$$\frac{\partial F_1^\alpha}{\partial x^\alpha} = \frac{\partial F_1^\alpha}{\partial \phi_I}\frac{\partial \phi_I}{\partial x^\alpha} + \frac{\partial F_1^\alpha}{\partial \Phi_I}\frac{\partial \Phi_I}{\partial x^\alpha} + \left.\frac{\partial F_1^\alpha}{\partial x^\alpha}\right|_{\text{expl}}. \tag{8}$$

The rightmost term denotes the sum over the *explicit* dependence of the generating function F_1^μ on the x^ν. Comparing the coefficients of Eqs. (7) and (8), we find the local coordinate representation of the field transformation rules that are induced by the generating function F_1^μ

$$\pi_I^\mu = \frac{\partial F_1^\mu}{\partial \phi_I}, \qquad \Pi_I^\mu = -\frac{\partial F_1^\mu}{\partial \Phi_I}, \qquad \mathcal{H}' = \mathcal{H} + \left.\frac{\partial F_1^\alpha}{\partial x^\alpha}\right|_{\text{expl}}. \tag{9}$$

The transformation rule for the Hamiltonian density implies that summation over α is to be performed. In contrast to the transformation rule for the Lagrangian density $\mathcal{L}$ of Eq. (7), the rule for the Hamiltonian density is determined by the *explicit* dependence of the generating function F_1^μ on the x^ν. Hence, if a generating function does not explicitly depend on the independent variables, x^ν, then the *value* of the Hamiltonian density is not changed under the particular canonical transformation emerging thereof.

Differentiating the transformation rule for π_I^μ with respect to Φ_J, and the rule for Π_J^μ with respect to ϕ_I, we obtain a symmetry relation between original and transformed fields

$$\frac{\partial \pi_I^\mu}{\partial \Phi_J} = \frac{\partial^2 F_1^\mu}{\partial \phi_I \partial \Phi_J} = -\frac{\partial \Pi_J^\mu}{\partial \phi_I}.$$

The emerging of symmetry relations is a characteristic feature of *canonical* transformations. As the symmetry relation directly follows from the second derivatives of the generating function, is does not apply for arbitrary transformations of the fields that do not follow from generating functions.

Generating functions of type $\mathbf{F}_2(\phi, \mathbf{\Pi}, x)$

The generating function of a canonical transformation can alternatively be expressed in terms of a function of the original fields ϕ_I and of the new *conjugate* fields Π_I^μ. To derive the pertaining transformation rules, we perform the covariant Legendre transformation

$$F_2^\mu(\phi, \mathbf{\Pi}, x) = F_1^\mu(\phi, \mathbf{\Phi}, x) + \Phi_J \Pi_J^\mu, \qquad \Pi_I^\mu = -\frac{\partial F_1^\mu}{\partial \Phi_I}. \tag{10}$$

By definition, the functions F_1^μ and F_2^μ agree with respect to their ϕ_I and x^μ dependencies

$$\frac{\partial F_2^\mu}{\partial \phi_I} = \frac{\partial F_1^\mu}{\partial \phi_I} = \pi_I^\mu, \qquad \left.\frac{\partial F_2^\alpha}{\partial x^\alpha}\right|_{\text{expl}} = \left.\frac{\partial F_1^\alpha}{\partial x^\alpha}\right|_{\text{expl}} = \mathcal{H}' - \mathcal{H}.$$

The variables ϕ_I and x^μ thus do not take part in the Legendre transformation from Eq. (10). Therefore, the two F_2^μ-related transformation rules coincide with the respective rules derived previously from F_1^μ. As F_1^μ does not depend on the Π_I^μ whereas F_2^μ does not depend on the the Φ_I, the new transformation rule thus follows from the derivative of F_2^μ with respect to Π_J^ν as

$$\frac{\partial F_2^\mu}{\partial \Pi_I^\nu} = \Phi_J \frac{\partial \Pi_J^\mu}{\partial \Pi_I^\nu} = \Phi_J \, \delta_{JI} \, \delta_\nu^\mu.$$

We thus end up with set of transformation rules

$$\pi_I^\mu = \frac{\partial F_2^\mu}{\partial \phi_I}, \qquad \Phi_I \, \delta_\nu^\mu = \frac{\partial F_2^\mu}{\partial \Pi_I^\nu}, \qquad \mathcal{H}' = \mathcal{H} + \left.\frac{\partial F_2^\alpha}{\partial x^\alpha}\right|_{\text{expl}}, \tag{11}$$

which is equivalent to the set (9) by virtue of the Legendre transformation (10) if the matrices $(\partial^2 F_1^\mu / \partial \phi_I \partial \Phi_J)$ are non-singular. From the second partial derivations of F_2^μ one immediately derives the symmetry relation

$$\frac{\partial \pi_I^\mu}{\partial \Pi_J^\nu} = \frac{\partial^2 F_2^\mu}{\partial \phi_I \partial \Pi_J^\nu} = \frac{\partial \Phi_J}{\partial \phi_I} \, \delta_\nu^\mu,$$

whose existence characterizes the transformation to be canonical.

Gauge theories as canonical transformations

Devising gauge theories in terms of canonical transformations turns out to be a particularly useful application of the canonical formalism in the realm of classical field theory. The systematic procedure to pursue is as follows:

1. Construct the generating function F_2^μ that defines the desired local transformation of the fields of the given covariant system Hamiltonian $\mathcal{H}$. If the given system is described in terms of a Lagrangian $\mathcal{L}$, the corresponding Hamiltonian $\mathcal{H}$ is obtained by a covariant Legendre transformation according to Eq. (4).
2. Calculate the divergence of F_2^μ to find the transformation rule for the Hamiltonian $\mathcal{H}$.
3. Introduce the appropriate gauge field Hamiltonian $\mathcal{H}_\mathrm{g}$ that is eligible to compensate the terms of the divergence of F_2^μ.
4. Derive the transformation rules for the gauge fields from the requirement that the amended Hamiltonian $\mathcal{H}_1 = \mathcal{H} + \mathcal{H}_\mathrm{g}$ be *form-invariant*.
5. Construct the amended generating function $\tilde{F}_2^\mu$ that defines the transformation of base fields *and* gauge fields.
6. Calculate the divergence of $\tilde{F}_2^\mu$ to find the transformation rule for the amended Hamiltonian $\mathcal{H}_1$.
7. Express the divergence of $\tilde{F}_2^\mu$ in terms of the physical fields and their conjugates making use of their transformation rules.
8. Provided that all terms come up *in pairs*, i.e., if they have the same form in the original and in the transformed field variables, this *uniquely* determines the form of the Hamiltonian $\mathcal{H}_2$ that is *locally form-invariant*.
9. Add the Hamiltonian $\mathcal{H}_\mathrm{kin}$ describing the kinetics of the *free* gauge fields. It must be ensured that $\mathcal{H}_\mathrm{kin}$ is also form-invariant under the given transformation rules to maintain the local form-invariance of the final Hamiltonian $\mathcal{H}_3 = \mathcal{H}_2 + \mathcal{H}_\mathrm{kin}$.
10. Optionally Legendre-transform the final Hamiltonian $\mathcal{H}_3$ to determine the corresponding locally gauge-invariant Lagrangian $\mathcal{L}_3$.

We will follow this procedure in the next section to work out a Lagrangian $\mathcal{L}_3$ that is form-invariant under an *inhomogeneous local gauge transformation*.

General inhomogeneous local gauge transformation

As a generalization of the homogeneous local U(N) gauge group, we now treat the corresponding *inhomogeneous* gauge group for the case of an N-tuple of fields ϕ_I.

External gauge fields

We consider a system consisting of an N-tuple ϕ of complex fields ϕ_I with $I = 1, \ldots, N$, and $\overline{\phi}$ its adjoint,

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}, \qquad \overline{\phi} = \left(\overline{\phi}_1 \cdots \overline{\phi}_N\right).$$

A general inhomogeneous linear transformation may be expressed in terms of a complex matrix $U(x) = \big(u_{IJ}(x)\big)$, $U^\dagger(x) = \big(\overline{u}_{IJ}(x)\big)$ and a vector $\varphi(x) = \big(\varphi_I(x)\big)$ that generally depend explicitly on the independent variables, x^μ, as

$$\begin{aligned} \phi \quad &= U\,\phi + \varphi, \qquad && \overline{\phi} = \overline{\phi}\,U^\dagger + \overline{\varphi} \\ \Phi_I \quad &= u_{IJ}\,\phi_J + \varphi_I, \qquad && \overline{\Phi}_I = \overline{\phi}_J\,\overline{u}_{JI} + \overline{\varphi}_I. \end{aligned} \tag{12}$$

With this notation, ϕ_I stands for a set of $I = 1, \ldots, N$ complex fields ϕ_I. In other words, U is supposed to define an isomorphism within the space of the ϕ_I, hence to linearly map the ϕ_I into objects of the same type. The quantities $\varphi_I(x)$ have the dimension of the base fields ϕ_I and define a *local* shifting transformation of the Φ_I in iso-space. Physically, this means that the system is now required to be form-invariant both under local unitary transformations in iso-space *and* under local variations of *background fields* $\varphi_I(x)$.

The transformation (12) follows from a generating function that—corresponding to $\mathcal{H}$—must be a real-valued function of the generally complex fields ϕ_I and their canonical conjugates, π_I^μ,

$$\begin{aligned} F_2^\mu(\phi, \overline{\phi}, \mathbf{\Pi}^\mu, \overline{\mathbf{\Pi}}^\mu, x) &= \overline{\mathbf{\Pi}}^\mu \left(U\,\phi + \varphi\right) + \left(\overline{\phi}\,U^\dagger + \overline{\varphi}\right) \mathbf{\Pi}^\mu \\ &= \overline{\Pi}_K^\mu \left(u_{KJ}\,\phi_J + \varphi_K\right) + \left(\overline{\phi}_K\,\overline{u}_{KJ} + \overline{\varphi}_J\right) \Pi_J^\mu. \end{aligned} \tag{13}$$

According to Eqs. (11) the set of transformation rules follows as

$$\begin{aligned} \overline{\pi}_I^\mu &= \frac{\partial F_2^\mu}{\partial \phi_I} = \overline{\Pi}_K^\mu u_{KJ} \delta_{JI}, & \overline{\Phi}_I \delta_\nu^\mu &= \frac{\partial F_2^\mu}{\partial \Pi_I^\nu} = \left(\overline{\phi}_K \overline{u}_{KJ} + \overline{\varphi}_J\right) \delta_\nu^\mu \delta_{JI} \\ \pi_I^\mu &= \frac{\partial F_2^\mu}{\partial \overline{\phi}_I} = \delta_{IK} \overline{u}_{KJ} \Pi_J^\mu, & \Phi_I \delta_\nu^\mu &= \frac{\partial F_2^\mu}{\partial \overline{\Pi}_I^\nu} = \delta_\nu^\mu \delta_{IK} \left(u_{KJ} \phi_J + \varphi_K\right). \end{aligned}$$

The complete set of transformation rules and their inverses then read in component notation

$$\begin{aligned} &\Phi_I = u_{IJ}\,\phi_J + \varphi_I, \overline{\Phi}_I = \overline{\phi}_J\,\overline{u}_{JI} + \overline{\varphi}_I, \Pi_I^\mu = u_{IJ}\,\pi_J^\mu, \overline{\Pi}_I^\mu = \overline{\pi}_J^\mu\,\overline{u}_{JI} \\ &\phi_I = \overline{u}_{IJ}\left(\Phi_J - \varphi_J\right), \overline{\phi}_I = \left(\overline{\Phi}_J - \overline{\varphi}_J\right) u_{JI}, \pi_I^\mu = \overline{u}_{IJ}\,\Pi_J^\mu, \overline{\pi}_I^\mu = \overline{\Pi}_J^\mu u_{JI}. \end{aligned} \tag{14}$$

We restrict ourselves to transformations that preserve the contraction $\overline{\pi}^\alpha \pi_\alpha$

$$\begin{aligned} \overline{\mathbf{\Pi}}^\alpha \mathbf{\Pi}_\alpha &= \overline{\pi}^\alpha\,U^\dagger U\,\pi_\alpha = \overline{\pi}^\alpha \pi_\alpha &\Longrightarrow \quad U^\dagger U = \mathbb{1} = U U^\dagger \\ \overline{\Pi}_I^\alpha \Pi_{I\alpha} &= \overline{\pi}_J^\alpha \overline{u}_{JI}\,u_{IK} \pi_{K\alpha} = \overline{\pi}_K^\alpha \pi_{K\alpha} &\Longrightarrow \quad \overline{u}_{JI}\,u_{IK} = \delta_{JK} = u_{JI}\,\overline{u}_{IK}. \end{aligned}$$

This means that $U^\dagger = U^{-1}$, hence that the matrix U is supposed to be *unitary*. As a unitary matrix, $U(x)$ is a member of the unitary group $\mathrm{U}(N)$

$$U^\dagger(x) = U^{-1}(x), \qquad |\det U(x)| = 1.$$

For $\det U(x) = +1$, the matrix $U(x)$ is a member of the special group $\mathrm{SU}(N)$.

We require the Hamiltonian density $\mathcal{H}$ to be *form-invariant* under the *global* gauge transformation (12), which is given for $U, \varphi = \text{const.}$, hence for all u_{IJ}, φ_I *not* depending on the independent variables, x^μ. Generally, if $U = U(x)$, $\varphi = \varphi(x)$, then the transformation (14) is referred to as a *local* gauge transformation. The transformation rule for the Hamiltonian is then determined by the explicitly x^μ-dependent terms of the generating function F_2^μ according to

$$\begin{aligned} \mathcal{H}' - \mathcal{H} &= \left.\frac{\partial F_2^\alpha}{\partial x^\alpha}\right|_{\text{expl}} \\ &= \overline{\Pi}_I^\alpha \left(\frac{\partial u_{IJ}}{\partial x^\alpha}\,\phi_J + \frac{\partial \varphi_I}{\partial x^\alpha}\right) + \left(\overline{\phi}_I \frac{\partial \overline{u}_{IJ}}{\partial x^\alpha} + \frac{\partial \overline{\varphi}_J}{\partial x^\alpha}\right) \Pi_J^\alpha \\ &= \overline{\pi}_K^\alpha\,\overline{u}_{KI} \left(\frac{\partial u_{IJ}}{\partial x^\alpha} \phi_J + \frac{\partial \varphi_I}{\partial x^\alpha}\right) + \left(\overline{\phi}_I \frac{\partial \overline{u}_{IJ}}{\partial x^\alpha} + \frac{\partial \overline{\varphi}_J}{\partial x^\alpha}\right) u_{JK} \pi_K^\alpha \\ &= \left(\overline{\pi}_K^\alpha\,\phi_J - \overline{\phi}_K \pi_J^\alpha\right) \overline{u}_{KI} \frac{\partial u_{IJ}}{\partial x^\alpha} + \overline{\pi}_I^\alpha \overline{u}_{IJ} \frac{\partial \varphi_J}{\partial x^\alpha} + \frac{\partial \overline{\varphi}_J}{\partial x^\alpha} u_{JI} \pi_I^\alpha. \end{aligned} \tag{15}$$

In the last step, the identity

$$\frac{\partial \overline{u}_{JI}}{\partial x^\mu}\,u_{IK} + \overline{u}_{JI}\,\frac{\partial u_{IK}}{\partial x^\mu} = 0$$

was inserted. If we want to set up a Hamiltonian $\mathcal{H}_1$ that is *form-invariant* under the *local*, hence x^μ-dependent transformation generated by (13), then we must compensate the additional terms (15) that emerge from the explicit x^μ-dependence of the generating function (13). The only way to achieve this is to *adjoin* the Hamiltonian $\mathcal{H}$ of our system with terms that correspond to (15) with regard to their dependence on the canonical variables, $\phi, \overline{\phi}, \pi^\mu, \overline{\pi}^\mu$. With a *unitary* matrix U, the u_{IJ}-dependent terms in Eq. (15) are *skew-Hermitian*,

$$\overline{\overline{u}_{KI}\,\frac{\partial u_{IJ}}{\partial x^\mu}} = \frac{\partial \overline{u}_{JI}}{\partial x^\mu}\,u_{IK} = -\overline{u}_{JI}\,\frac{\partial u_{IK}}{\partial x^\mu},$$

$$\overline{\frac{\partial u_{KI}}{\partial x^\mu}\,\overline{u}_{IJ}} = u_{JI}\frac{\partial \overline{u}_{IK}}{\partial x^\mu} = -\frac{\partial u_{JI}}{\partial x^\mu}\overline{u}_{IK}, \tag{16}$$

or in matrix notation

$$\left(U^\dagger\frac{\partial U}{\partial x^\mu}\right)^\dagger = \frac{\partial U^\dagger}{\partial x^\mu}U = -U^\dagger\frac{\partial U}{\partial x^\mu}, \qquad \left(\frac{\partial U}{\partial x^\mu}U^\dagger\right)^\dagger = U\frac{\partial U^\dagger}{\partial x^\mu} = -\frac{\partial U}{\partial x^\mu}U^\dagger.$$

The $\overline{u}_{KI}\partial u_{IJ}/\partial x^\mu$-dependent terms in Eq. (15) can thus be compensated by a *Hermitian* matrix $(\boldsymbol{a}_{KJ})$ of "4-vector gauge fields", with each off-diagonal matrix element, $\boldsymbol{a}_{KJ}$, $K \neq J$, a complex 4-vector field with components $a_{KJ\mu}$, $\mu = 0, \ldots, 3$

$$\overline{u}_{KI}\frac{\partial u_{IJ}}{\partial x^\mu} \quad \leftrightarrow \quad a_{KJ\mu}, \qquad a_{KJ\mu} = \overline{a}_{KJ\mu} = a^*_{JK\mu}.$$

Correspondingly, the term proportional to $\overline{u}_{IJ}\partial\varphi_J/\partial x^\mu$ is compensated by the μ-components $M_{IJ}b_{J\mu}$ of a vector $M_{IJ}\,\boldsymbol{b}_J$ of 4-vector gauge fields,

$$\overline{u}_{IJ}\frac{\partial\varphi_J}{\partial x^\mu} \quad \leftrightarrow \quad M_{IJ}b_{J\mu}, \qquad \frac{\partial\overline{\varphi}_J}{\partial x^\mu}u_{JI} \quad \leftrightarrow \quad \overline{b}_{J\mu}M_{IJ}.$$

The term proportional to $\partial\overline{\varphi}_J/\partial x\, u_{JI}$ is then compensated by the adjoint vector $\overline{\boldsymbol{b}}_J M_{IJ}$. The dimension of the constant real matrix M is $[M] = L^{-1}$ and thus has the natural dimension of mass. The given system Hamiltonian $\mathcal{H}$ must be amended by a Hamiltonian $\mathcal{H}_\mathrm{a}$ of the form

$$\mathcal{H}_1 = \mathcal{H} + \mathcal{H}_\mathrm{a}, \quad \mathcal{H}_\mathrm{a} = \mathrm{i}g\left(\overline{\pi}^\alpha_K\phi_J - \overline{\phi}_K\pi^\alpha_J\right)a_{KJ\alpha} + \overline{\pi}^\alpha_I M_{IJ}b_{J\alpha} + \overline{b}_{J\alpha}M_{IJ}\pi^\alpha_I \tag{17}$$

in order for $\mathcal{H}_1$ to be *form-invariant* under the canonical transformation that is defined by the explicitly x^μ-dependent generating function from Eq. (13). With a real coupling constant g, the "gauge Hamiltonian" $\mathcal{H}_\mathrm{a}$ is thus real. Submitting the amended Hamiltonian $\mathcal{H}_1$ to the canonical transformation generated by Eq. (13), the new Hamiltonian $\mathcal{H}'_1$ emerges as

$$\begin{aligned}
\mathcal{H}'_1 &= \mathcal{H}_1 + \left.\tfrac{\partial F^\alpha_2}{\partial x^\alpha}\right|_\mathrm{expl} = \mathcal{H} + \mathcal{H}_\mathrm{a} + \left.\tfrac{\partial F^\alpha_2}{\partial x^\alpha}\right|_\mathrm{expl} \\
&= \mathcal{H} + \left(\overline{\pi}^\alpha_K\phi_J - \overline{\phi}_K\pi^\alpha_J\right)\left(\mathrm{i}g\,a_{KJ\alpha} + \overline{u}_{KI}\tfrac{\partial u_{IJ}}{\partial x^\alpha}\right) \\
&\quad + \overline{\pi}^\alpha_I\left(M_{IJ}b_{J\alpha} + \overline{u}_{IJ}\tfrac{\partial\varphi_J}{\partial x^\alpha}\right) + \left(\overline{b}_{J\alpha}M_{IJ} + \tfrac{\partial\overline{\varphi}_J}{\partial x^\alpha}u_{JI}\right)\pi^\alpha_I \\
&\stackrel{!}{=} \mathcal{H}' + \mathrm{i}g\left(\overline{\Pi}^\alpha_K\Phi_J - \overline{\Phi}_K\Pi^\alpha_J\right)A_{KJ\alpha} + \overline{\Pi}^\alpha_I M_{IJ}B_{J\alpha} + \overline{B}_{J\alpha}M_{IJ}\Pi^\alpha_I,
\end{aligned}$$

with the $A_{IJ\mu}$ and $B_{I\mu}$ defining the gauge field components of the transformed system. The *form* of the system Hamiltonian $\mathcal{H}_1$ is thus maintained under the canonical transformation,

$$\begin{aligned}
\mathcal{H}'_1 &= \mathcal{H}' + \mathcal{H}'_\mathrm{a}, \\
\mathcal{H}'_\mathrm{a} &= \mathrm{i}g\left(\overline{\Pi}^\alpha_K\Phi_J - \overline{\Phi}_K\Pi^\alpha_J\right)A_{KJ\alpha} + \overline{\Pi}^\alpha_I M_{IJ}B_{J\alpha} + \overline{B}_{J\alpha}M_{IJ}\Pi^\alpha_I,
\end{aligned} \tag{18}$$

provided that the given system Hamiltonian $\mathcal{H}$ is form-invariant under the corresponding *global* gauge transformation (14). In other words, we suppose the given system Hamiltonian $\mathcal{H}(\phi, \overline{\phi}, \pi^\mu, \overline{\pi}^\mu, x)$ to remain form-invariant if it is expressed in terms of the transformed fields,

$$\mathcal{H}'(\phi, \overline{\phi}, \mathbf{\Pi}^\mu, \overline{\mathbf{\Pi}}^\mu, x) \overset{\text{global GT}}{=} \mathcal{H}(\phi, \overline{\phi}, \pi^\mu, \overline{\pi}^\mu, x).$$

Replacing the transformed base fields by the original ones according to Eqs. (14), the gauge fields must satisfy the condition

$$\begin{aligned}
&\left(\overline{\pi}^\alpha_K \phi_J - \overline{\phi}_K \pi^\alpha_J\right)\left(\mathrm{i}g\, a_{KJ\alpha} + \overline{u}_{KI}\tfrac{\partial u_{LJ}}{\partial x^\alpha}\right)\\
&+ \overline{\pi}^\alpha_I\left(M_{IJ} b_{J\alpha} + \overline{u}_{IJ}\tfrac{\partial \varphi_J}{\partial x^\alpha}\right) + \left(\overline{b}_{J\alpha} M_{IJ} + \tfrac{\partial \overline{\varphi}_J}{\partial x^\alpha} u_{JI}\right)\pi^\alpha_I\\
&= \mathrm{i}g\left(\overline{\pi}^\alpha_I \overline{u}_{IK} u_{JL}\phi_L + \overline{\pi}^\alpha_I \overline{u}_{IK}\varphi_J - \overline{\phi}_L \overline{u}_{LK} u_{JI}\pi^\alpha_I - \overline{\varphi}_K u_{JI}\pi^\alpha_I\right) A_{KJ\alpha}\\
&+ \overline{\pi}^\alpha_I \overline{u}_{IK} M_{KJ} B_{J\alpha} + \overline{B}_{J\alpha} M_{KJ} u_{KI}\pi^\alpha_I,
\end{aligned}$$

which yields with Eqs. (14) the following inhomogeneous transformation rules for the gauge fields $\boldsymbol{a}_{KJ}$, $\boldsymbol{b}_J$, and $\overline{\boldsymbol{b}}_J$ by comparing the coefficients that are associated with the independent dynamical variables π^μ_I, $\overline{\pi}^\mu_I$, $\overline{\pi}^\mu_I \phi_J$, and $\overline{\phi}_J \pi^\mu_I$

$$\begin{aligned}
A_{KJ\mu} &= u_{KL}\, a_{LI\mu}\, \overline{u}_{IJ} + \tfrac{1}{\mathrm{i}g}\tfrac{\partial u_{KI}}{\partial x^\mu}\,\overline{u}_{IJ}\\
B_{J\mu} &= \tilde{M}_{JI}\left(u_{IK} M_{KL} b_{L\mu} - \mathrm{i}g\, A_{IK\mu}\varphi_K + \tfrac{\partial \varphi_I}{\partial x^\mu}\right)\\
\overline{B}_{J\mu} &= \left(\overline{b}_{L\mu} M_{KL}\overline{u}_{KI} + \mathrm{i}g\,\overline{\varphi}_K A_{KI\mu} + \tfrac{\partial \overline{\varphi}_I}{\partial x^\mu}\right)\tilde{M}_{JI}.
\end{aligned} \tag{19}$$

Herein, $\tilde{M}$ denotes the inverse matrix of M, hence $\tilde{M}_{KJ} M_{JI} = M_{KJ}\tilde{M}_{JI} = \delta_{KI}$. We observe that for any type of canonical field variables ϕ_I and for any Hamiltonian system $\mathcal{H}$, the transformation of both the matrix $\boldsymbol{a}_{IJ}$ as well as the vector $\boldsymbol{b}_I$ of 4-vector gauge fields is uniquely determined according to Eq. (19) by the unitary matrix $U(x)$ and the translation vector $\varphi(x)$ that determine the *local* transformation of the N base fields ϕ. In a more concise matrix notation, Eqs. (19) are

$$\begin{aligned}
\boldsymbol{A}_\mu &= U\,\boldsymbol{a}_\mu\, U^\dagger + \tfrac{1}{\mathrm{i}g}\tfrac{\partial U}{\partial x^\mu}\, U^\dagger\\
M\boldsymbol{B}_\mu &= UM\,\boldsymbol{b}_\mu - \mathrm{i}g\,\boldsymbol{A}_\mu\varphi + \tfrac{\partial \varphi}{\partial x^\mu}\\
\overline{\boldsymbol{B}}_\mu M^T &= \overline{\boldsymbol{b}}_\mu M^T U^\dagger + \mathrm{i}g\,\overline{\varphi}\,\boldsymbol{A}_\mu + \tfrac{\partial \overline{\varphi}}{\partial x^\mu}.
\end{aligned} \tag{20}$$

Inserting the transformation rules for the base fields, $\phi = U\phi + \varphi$ and $\overline{\phi} = \overline{\phi}\,U^\dagger + \overline{\varphi}$ into Eqs. (20), we immediately find the *homogeneous* transformation conditions

$$\begin{aligned}
\frac{\partial \phi}{\partial x^\mu} - \mathrm{i}g\,\boldsymbol{A}_\mu\phi - M\boldsymbol{B}_\mu &= U\left(\tfrac{\partial \phi}{\partial x^\mu} - \mathrm{i}g\,\boldsymbol{a}_\mu\phi - M\boldsymbol{b}_\mu\right)\\
\frac{\partial \overline{\phi}}{\partial x^\mu} + \mathrm{i}g\,\overline{\phi}\boldsymbol{A}_\mu - \overline{\boldsymbol{B}}_\mu M^T &= \left(\tfrac{\partial \overline{\phi}}{\partial x^\mu} + \mathrm{i}g\,\overline{\phi}\boldsymbol{a}_\mu - \overline{\boldsymbol{b}}_\mu M^T\right)U^\dagger.
\end{aligned}$$

We identify the "amended" partial derivatives as the "covariant derivative" that defines the "minimum coupling rule" for our inhomogeneous gauge transformation. It reduces to the conventional minimum coupling rule for the homogeneous gauge transformation, hence for $\varphi \equiv 0, M \equiv 0$.

Including the gauge field dynamics

With the knowledge of the required transformation rules for the gauge fields from Eq. (19), it is now possible to redefine the generating function (13) to also describe the gauge field transformations. This simultaneously defines the transformations of the canonical conjugates, $p^{\mu\nu}_{JK}$ and $q^{\mu\nu}_J$, of the gauge fields $a_{JK\mu}$ and $b_{J\mu}$, respectively. Furthermore, the redefined generating function yields additional terms in the transformation rule for the Hamiltonian. Of course, in order for the Hamiltonian to be invariant under local gauge transformations, the additional terms must be invariant as well. The transformation rules for the base fields ϕ_I and the gauge fields

$\boldsymbol{a}_{IJ}, \boldsymbol{b}_I$ (Eq. (19)) can be regarded as a canonical transformation that emerges from an explicitly x^μ-dependent and real-valued generating function vector of type $\tilde{F}_2^\mu = \tilde{F}_2^\mu(\phi, \overline{\phi}, \boldsymbol{\Pi}, \overline{\boldsymbol{\Pi}}, \boldsymbol{a}, \boldsymbol{P}, \boldsymbol{b}, \overline{\boldsymbol{b}}, \boldsymbol{Q}, \overline{\boldsymbol{Q}}, x)$,

$$\begin{aligned}
\tilde{F}_2^\mu = {} & \overline{\Pi}_K^\mu \left(u_{KJ}\, \phi_J + \varphi_K \right) + \left(\overline{\phi}_K\, \overline{u}_{KJ} + \overline{\varphi}_J \right) \Pi_J^\mu \\
& + \left(P_{JK}^{\alpha\mu} + \mathrm{i}g\, \tilde{M}_{LJ} Q_L^{\alpha\mu} \overline{\varphi}_K - \mathrm{i}g\, \varphi_J \overline{Q}_L^{\alpha\mu} \tilde{M}_{LK} \right) \\
& \qquad \left(u_{KN}\, a_{NI\alpha}\, \overline{u}_{IJ} + \frac{1}{\mathrm{i}g} \frac{\partial u_{KI}}{\partial x^\alpha}\, \overline{u}_{IJ} \right) \\
& + \overline{Q}_L^{\alpha\mu} \tilde{M}_{LK} \left(u_{KI} M_{IJ} b_{J\alpha} + \frac{\partial \varphi_K}{\partial x^\alpha} \right) + \\
& \qquad \left(\overline{b}_{K\alpha} M_{IK} \overline{u}_{IJ} + \frac{\partial \overline{\varphi}_J}{\partial x^\alpha} \right) \tilde{M}_{LJ} Q_L^{\alpha\mu}.
\end{aligned} \tag{21}$$

With the first line of (21) matching Eq. (13), the transformation rules for canonical variables $\phi, \overline{\phi}$ and their conjugates, $\pi^\mu, \overline{\pi}^\mu$, agree with those from Eqs. (14). The rules for the gauge fields $A_{KJ\alpha}$, $B_{K\alpha}$, and $\overline{B}_{K\alpha}$ emerge as

$$\begin{aligned}
A_{KJ\alpha}\, \delta_\nu^\mu = \frac{\partial \tilde{F}_2^\mu}{\partial P_{JK}^{\alpha\nu}} &= \delta_\nu^\mu \left(u_{KN}\, a_{NI\alpha}\, \overline{u}_{IJ} + \frac{1}{\mathrm{i}g} \frac{\partial u_{KI}}{\partial x^\alpha} \overline{u}_{IJ} \right) \\
B_{L\alpha}\, \delta_\nu^\mu = \frac{\partial \tilde{F}_2^\mu}{\partial \overline{Q}_L^{\alpha\nu}} &= \\
\delta_\nu^\mu \tilde{M}_{LK} \left[u_{KI} M_{IJ} b_{J\alpha} + \frac{\partial \varphi_K}{\partial x^\alpha} - \left(\mathrm{i}g\, u_{KN}\, a_{NI\alpha}\, \overline{u}_{IJ} + \frac{\partial u_{KI}}{\partial x^\alpha} \overline{u}_{IJ} \right) \varphi_J \right] & \\
= \delta_\nu^\mu \tilde{M}_{LK} \left(u_{KI} M_{IJ} b_{J\alpha} + \frac{\partial \varphi_K}{\partial x^\alpha} - \mathrm{i}g\, A_{KJ\alpha} \varphi_J \right) & \\
\overline{B}_{L\alpha}\, \delta_\nu^\mu = \frac{\partial \tilde{F}_2^\mu}{\partial Q_L^{\alpha\nu}} &= \\
\delta_\nu^\mu \left[\overline{b}_{K\alpha} M_{IK} \overline{u}_{IJ} + \frac{\partial \overline{\varphi}_J}{\partial x^\alpha} + \overline{\varphi}_K \left(\mathrm{i}g\, u_{KN}\, a_{NI\alpha}\, \overline{u}_{IJ} + \frac{\partial u_{KI}}{\partial x^\alpha} \overline{u}_{IJ} \right) \right] \tilde{M}_{LJ} & \\
= \delta_\nu^\mu \left(\overline{b}_{K\alpha} M_{IK} \overline{u}_{IJ} + \frac{\partial \overline{\varphi}_J}{\partial x^\alpha} + \mathrm{i}g\, \overline{\varphi}_K\, A_{KJ\alpha} \right) \tilde{M}_{LJ}, &
\end{aligned}$$

which obviously coincide with Eqs. (19) as the generating function (21) was devised accordingly. The transformation of the conjugate momentum fields is obtained from the generating function (21) as

$$\begin{aligned}
q_J^{\nu\mu} = \frac{\partial \tilde{F}_2^\mu}{\partial \overline{b}_{J\nu}} &= M_{IJ}\, \overline{u}_{IK}\, \tilde{M}_{LK}\, Q_L^{\nu\mu}, \quad \tilde{M}_{KJ} Q_K^{\nu\mu} = u_{JI}\, \tilde{M}_{KI}\, q_K^{\nu\mu} \\
\overline{q}_J^{\nu\mu} = \frac{\partial \tilde{F}_2^\mu}{\partial b_{J\nu}} &= \overline{Q}_L^{\nu\mu} \tilde{M}_{LK}\, u_{KI}\, M_{IJ}, \quad \overline{Q}_K^{\nu\mu} \tilde{M}_{KJ} = \overline{q}_K^{\nu\mu} \tilde{M}_{KI}\, \overline{u}_{IJ} \\
p_{IN}^{\nu\mu} = \frac{\partial \tilde{F}_2^\mu}{\partial a_{NI\nu}} &= \overline{u}_{IJ} \left(P_{JK}^{\nu\mu} + \mathrm{i}g\, \tilde{M}_{LJ} Q_L^{\nu\mu}\, \overline{\varphi}_K - \mathrm{i}g\, \varphi_J\, \overline{Q}_L^{\nu\mu} \tilde{M}_{LK} \right) u_{KN} \\
&= \overline{u}_{IJ} \left(P_{JK}^{\nu\mu} + \mathrm{i}g\, \tilde{M}_{LJ} Q_L^{\nu\mu}\, \overline{\Phi}_K - \mathrm{i}g\, \Phi_J\, \overline{Q}_L^{\nu\mu} \tilde{M}_{LK} \right) u_{KN} \\
& \qquad - \mathrm{i}g\, \tilde{M}_{LI} q_L^{\nu\mu}\, \overline{\phi}_N \;+\; \mathrm{i}g\, \phi_I\, \overline{q}_L^{\nu\mu} \tilde{M}_{LN}.
\end{aligned} \tag{22}$$

Thus, the expression

$$\begin{aligned}
& p_{IN}^{\nu\mu} + \mathrm{i}g\, \tilde{M}_{LI} q_L^{\nu\mu}\, \overline{\phi}_N - \mathrm{i}g\, \phi_I\, \overline{q}_L^{\nu\mu} \tilde{M}_{LN} \\
& = \overline{u}_{IJ} \left(P_{JK}^{\nu\mu} + \mathrm{i}g\, \tilde{M}_{LJ} Q_L^{\nu\mu}\, \overline{\Phi}_K - \mathrm{i}g\, \Phi_J\, \overline{Q}_L^{\nu\mu} \tilde{M}_{LK} \right) u_{KN}
\end{aligned} \tag{23}$$

transforms *homogeneously* under the gauge transformation generated by Eq. (21). The same homogeneous transformation law holds for the expression

$$\begin{aligned} f_{IJ\mu\nu} &= \tfrac{\partial a_{IJ\nu}}{\partial x^\mu} - \tfrac{\partial a_{IJ\mu}}{\partial x^\nu} + \mathrm{i}g\,(a_{IK\nu}a_{KJ\mu} - a_{IK\mu}a_{KJ\nu}) \\ &= \overline{u}_{IK}\,F_{KL\mu\nu}\,u_{LJ} \\ F_{IJ\mu\nu} &= \tfrac{\partial A_{IJ\nu}}{\partial x^\mu} - \tfrac{\partial A_{IJ\mu}}{\partial x^\nu} + \mathrm{i}g\,(A_{IK\nu}A_{KJ\mu} - A_{IK\mu}A_{KJ\nu})\,, \end{aligned} \tag{24}$$

which directly follows from the transformation rule (19) for the gauge fields $a_{IJ\mu}$. Making use of the initially defined mapping of the base fields (12), the transformation rule (19) for the gauge fields $b_{K\mu}, \overline{b}_{K\mu}$ is converted into

$$\begin{aligned} \frac{\partial \Phi_J}{\partial x^\mu} - \mathrm{i}g\,A_{JK\mu}\Phi_K - M_{JK}B_{K\mu} = \\ u_{JL}\left(\frac{\partial \phi_L}{\partial x^\mu} - \mathrm{i}g\,a_{LK\mu}\phi_K - M_{LK}b_{K\mu}\right) \\ \frac{\partial \overline{\Phi}_J}{\partial x^\mu} + \mathrm{i}g\,\overline{\Phi}_K A_{KJ\mu} - \overline{B}_{K\mu}M_{JK} = \\ \left(\frac{\partial \overline{\phi}_L}{\partial x^\mu} + \mathrm{i}g\,\overline{\phi}_K a_{KL\mu} - \overline{b}_{K\mu}M_{LK}\right)\overline{u}_{LJ}. \end{aligned} \tag{25}$$

The above transformation rules can also be expressed more clearly in matrix notation

$$\begin{aligned} \boldsymbol{q}^{\nu\mu} &= M^T U^\dagger \tilde{M}^T \boldsymbol{Q}^{\nu\mu}, \qquad \tilde{M}^T\boldsymbol{Q}^{\nu\mu} = U\tilde{M}^T\boldsymbol{q}^{\nu\mu} \\ \overline{\boldsymbol{q}}^{\nu\mu} &= \overline{\boldsymbol{Q}}^{\nu\mu}\tilde{M}\,U\,M, \qquad \overline{\boldsymbol{Q}}^{\nu\mu}\tilde{M} = \overline{\boldsymbol{q}}^{\nu\mu}\tilde{M}\,U^\dagger \\ \boldsymbol{p}^{\nu\mu} &= U^\dagger\left(\boldsymbol{P}^{\nu\mu} + \mathrm{i}g\,\tilde{M}^T\boldsymbol{Q}^{\nu\mu}\otimes\overline{\varphi} - \mathrm{i}g\,\varphi\otimes\overline{\boldsymbol{Q}}^{\nu\mu}\tilde{M}\right)U \\ \boldsymbol{f}_{\mu\nu} &= U^\dagger\,\mathbf{F}_{\mu\nu}\,U, \qquad \boldsymbol{f}_{\mu\nu} = \tfrac{\partial \boldsymbol{a}_\nu}{\partial x^\mu} - \tfrac{\partial \boldsymbol{a}_\mu}{\partial x^\nu} + \mathrm{i}g\,(\boldsymbol{a}_\nu\boldsymbol{a}_\mu - \boldsymbol{a}_\mu\boldsymbol{a}_\nu) \end{aligned} \tag{26}$$

and

$$\begin{aligned} \frac{\partial \phi}{\partial x^\mu} - \mathrm{i}g\,\boldsymbol{A}_\mu\phi - M\boldsymbol{B}_\mu = U\left(\frac{\partial \phi}{\partial x^\mu} - \mathrm{i}g\,\boldsymbol{a}_\mu\phi - M\,\boldsymbol{b}_\mu\right) \\ \frac{\partial \overline{\phi}}{\partial x^\mu} + \mathrm{i}g\,\overline{\phi}\boldsymbol{A}_\mu - \overline{\boldsymbol{B}}_\mu M^T = \left(\frac{\partial \overline{\phi}}{\partial x^\mu} + \mathrm{i}g\,\overline{\phi}\,\boldsymbol{a}_\mu - \overline{\boldsymbol{b}}_\mu M^T\right)U^\dagger \\ \boldsymbol{P}^{\nu\mu} + \mathrm{i}g\,\tilde{M}^T\boldsymbol{Q}^{\nu\mu}\otimes\overline{\phi} - \mathrm{i}g\,\phi\otimes\overline{\boldsymbol{Q}}^{\nu\mu}\tilde{M} = \\ U\left(\boldsymbol{p}^{\nu\mu} + \mathrm{i}g\,\tilde{M}^T\boldsymbol{q}^{\nu\mu}\otimes\overline{\phi} - \mathrm{i}g\,\phi\otimes\overline{\boldsymbol{q}}^{\nu\mu}\tilde{M}\right)U^\dagger. \end{aligned}$$

It remains to work out the difference of the Hamiltonians that are submitted to the canonical transformation generated by (21). Hence, according to the general rule from Eq. (11), we must calculate the divergence of the explicitly x^μ-dependent terms of $\tilde{F}_2^\mu$

$$\begin{aligned} \left.\frac{\partial \tilde{F}_2^\alpha}{\partial x^\alpha}\right|_{\mathrm{expl}} = \overline{\Pi}_K^\alpha\left(\frac{\partial u_{KJ}}{\partial x^\alpha}\,\phi_J + \frac{\partial \varphi_K}{\partial x^\alpha}\right) + \left(\overline{\phi}_K\,\frac{\partial \overline{u}_{KJ}}{\partial x^\alpha} + \frac{\partial \overline{\varphi}_J}{\partial x^\alpha}\right)\Pi_J^\alpha \\ + \left(P_{JK}^{\alpha\beta} + \mathrm{i}g\,\tilde{M}_{LJ}Q_L^{\alpha\beta}\,\overline{\varphi}_K - \mathrm{i}g\,\varphi_J\,\overline{Q}_L^{\alpha\beta}\tilde{M}_{LK}\right) \\ \cdot\left(\frac{\partial u_{KN}}{\partial x^\beta}a_{NI\alpha}\overline{u}_{IJ} + u_{KN}a_{NI\alpha}\frac{\partial \overline{u}_{IJ}}{\partial x^\beta} + \frac{1}{\mathrm{i}g}\frac{\partial u_{KI}}{\partial x^\alpha}\frac{\partial \overline{u}_{IJ}}{\partial x^\beta} + \frac{1}{\mathrm{i}g}\frac{\partial^2 u_{KI}}{\partial x^\alpha\partial x^\beta}\overline{u}_{IJ}\right) \\ + \left(\tilde{M}_{LJ}Q_L^{\alpha\beta}\frac{\partial \overline{\varphi}_K}{\partial x^\beta} - \frac{\partial \varphi_J}{\partial x^\beta}\,\overline{Q}_L^{\alpha\beta}\tilde{M}_{LK}\right)\left(\mathrm{i}g\,u_{KN}\,a_{NI\alpha}\,\overline{u}_{IJ} + \frac{\partial u_{KI}}{\partial x^\alpha}\,\overline{u}_{IJ}\right) \\ + \overline{Q}_L^{\alpha\beta}\tilde{M}_{LK}\left(\frac{\partial u_{KI}}{\partial x^\beta}M_{IJ}b_{J\alpha} + \frac{\partial^2 \varphi_K}{\partial x^\alpha\partial x^\beta}\right) + \\ \left(\overline{b}_{K\alpha}M_{IK}\frac{\partial \overline{u}_{IJ}}{\partial x^\beta} + \frac{\partial^2 \overline{\varphi}_J}{\partial x^\alpha\partial x^\beta}\right)\tilde{M}_{LJ}Q_L^{\alpha\beta}. \end{aligned}$$

(27)

We are now going to express all u_{IJ}- and φ_K-dependencies in (27) in terms of the field variables making use of the canonical transformation rules. To this end, the constituents of Eq. (27) are split into three blocks. The $\mathbf{\Pi}$-dependent terms of can be converted this way by means of the transformation rules (14) and (19)

$$\begin{aligned}
&\overline{\Pi}^\alpha_K \left(\tfrac{\partial u_{KJ}}{\partial x^\alpha}\, \phi_J + \tfrac{\partial \varphi_K}{\partial x^\alpha} \right) + \left(\overline{\phi}_K\, \tfrac{\partial \overline{u}_{KJ}}{\partial x^\alpha} + \tfrac{\partial \overline{\varphi}_J}{\partial x^\alpha} \right) \Pi^\alpha_J \\
&= \overline{\Pi}^\alpha_K \left(\tfrac{\partial u_{KJ}}{\partial x^\alpha}\, \overline{u}_{JI}(\Phi_I - \varphi_I) + \tfrac{\partial \varphi_K}{\partial x^\alpha} \right) + \left(\left(\overline{\Phi}_I - \overline{\varphi}_I \right) u_{IK}\, \tfrac{\partial \overline{u}_{KJ}}{\partial x^\alpha} + \tfrac{\partial \overline{\varphi}_J}{\partial x^\alpha} \right) \Pi^\alpha_J \\
&= \mathrm{i}g \left(\overline{\Pi}^\alpha_K \Phi_J - \overline{\Phi}_K \Pi^\alpha_J \right) A_{KJ\alpha} + \overline{\Pi}^\alpha_K M_{KJ} B_{J\alpha} + \overline{B}_{K\alpha} M_{JK} \Pi^\alpha_J \\
&\quad - \mathrm{i}g \left(\overline{\pi}^\alpha_K \phi_J - \overline{\phi}_K \pi^\alpha_J \right) a_{KJ\alpha} - \left(\overline{\pi}^\alpha_K M_{KJ} b_{J\alpha} + \overline{b}_{K\alpha} M_{JK} \pi^\alpha_J \right).
\end{aligned}$$

(28)

The second derivative terms in Eq. (27) are *symmetric* in the indices α and β. If we split $P^{\alpha\beta}_{JK}$ and $Q^{\alpha\beta}_J$ into a symmetric $P^{(\alpha\beta)}_{JK}, Q^{(\alpha\beta)}_J$ and a skew-symmetric parts $P^{[\alpha\beta]}_{JK}, P^{[\alpha\beta]}_J$ in α and β

$$\begin{aligned}
P^{\alpha\beta}_{JK} &= P^{(\alpha\beta)}_{JK} + P^{[\alpha\beta]}_{JK}, P^{[\alpha\beta]}_{JK} = \tfrac{1}{2} \left(P^{\alpha\beta}_{JK} - P^{\beta\alpha}_{JK} \right), P^{(\alpha\beta)}_{JK} = \tfrac{1}{2} \left(P^{\alpha\beta}_{JK} + P^{\beta\alpha}_{JK} \right) \\
Q^{\alpha\beta}_J &= Q^{(\alpha\beta)}_J + Q^{[\alpha\beta]}_J, Q^{[\alpha\beta]}_J = \tfrac{1}{2} \left(Q^{\alpha\beta}_J - Q^{\beta\alpha}_J \right), Q^{(\alpha\beta)}_J = \tfrac{1}{2} \left(Q^{\alpha\beta}_J + Q^{\beta\alpha}_J \right),
\end{aligned}$$

then the second derivative terms in Eq. (27) vanish for $P^{[\alpha\beta]}_{JK}$ and $Q^{[\alpha\beta]}_J$,

$$P^{[\alpha\beta]}_{JK} \frac{\partial^2 u_{KI}}{\partial x^\alpha \partial x^\beta} = 0, \qquad \frac{\partial^2 \overline{\varphi}_J}{\partial x^\alpha \partial x^\beta} Q^{[\alpha\beta]}_J = 0, \qquad \overline{Q}^{[\alpha\beta]}_K \frac{\partial^2 \varphi_K}{\partial x^\alpha \partial x^\beta} = 0.$$

By inserting the transformation rules for the gauge fields from Eqs. (19), the remaining terms of (27) for the skew-symmetric part of $P^{\alpha\beta}_{JK}$ are converted into

$$\begin{aligned}
&\left(P^{[\alpha\beta]}_{JK} + \mathrm{i}g\, \tilde{M}_{LJ} Q^{[\alpha\beta]}_L\, \overline{\varphi}_K - \mathrm{i}g\, \varphi_J\, \overline{Q}^{[\alpha\beta]}_L \tilde{M}_{LK} \right) \\
&\cdot \left(\tfrac{\partial u_{KN}}{\partial x^\beta} a_{NI\alpha} \overline{u}_{IJ} + u_{KN} a_{NI\alpha} \tfrac{\partial \overline{u}_{IJ}}{\partial x^\beta} + \tfrac{1}{\mathrm{i}g} \tfrac{\partial u_{KI}}{\partial x^\alpha} \tfrac{\partial \overline{u}_{IJ}}{\partial x^\beta} \right) \\
&\quad + \left(\tilde{M}_{LJ} Q^{[\alpha\beta]}_L\, \tfrac{\partial \overline{\varphi}_K}{\partial x^\beta} - \tfrac{\partial \varphi_J}{\partial x^\beta} \overline{Q}^{[\alpha\beta]}_L \tilde{M}_{LK} \right) \mathrm{i}g\, A_{KJ\alpha} \\
&+ \overline{Q}^{[\alpha\beta]}_L \tilde{M}_{LK} \tfrac{\partial u_{KI}}{\partial x^\beta} M_{IJ} b_{J\alpha} + \overline{b}_{J\alpha} M_{IJ} \tfrac{\partial \overline{u}_{IK}}{\partial x^\beta} \tilde{M}_{LK} Q^{[\alpha\beta]}_L \\
&= -\tfrac{1}{2}\mathrm{i}g\, P^{\alpha\beta}_{JK} \left(A_{KI\alpha} A_{IJ\beta} - A_{KI\beta} A_{IJ\alpha} \right) \\
&+ \tfrac{1}{2}\mathrm{i}g \left(\overline{B}_{J\beta} M_{KJ} A_{KI\alpha} \tilde{M}_{IL} - \overline{B}_{J\alpha} M_{KJ} A_{KI\beta} \tilde{M}_{IL} \right) Q^{\alpha\beta}_L \\
&- \tfrac{1}{2}\mathrm{i}g\, \overline{Q}^{\alpha\beta}_L \left(\tilde{M}_{LI} A_{IK\alpha} M_{KJ} B_{J\beta} - \tilde{M}_{LI} A_{IK\beta} M_{KJ} B_{J\alpha} \right) \\
&\quad + \tfrac{1}{2}\mathrm{i}g\, p^{\alpha\beta}_{JK} \left(a_{KI\alpha} a_{IJ\beta} - a_{KI\beta} a_{IJ\alpha} \right) \\
&- \tfrac{1}{2}\mathrm{i}g \left(\overline{b}_{J\beta} M_{KJ} a_{KI\alpha} \tilde{M}_{IL} - \overline{b}_{J\alpha} M_{KJ} a_{KI\beta} \tilde{M}_{LI} \right) q^{\alpha\beta}_L \\
&+ \tfrac{1}{2}\mathrm{i}g\, \overline{q}^{\alpha\beta}_L \left(\tilde{M}_{LI} a_{IK\alpha} M_{KJ} b_{J\beta} - \tilde{M}_{LI} a_{IK\beta} M_{KJ} b_{J\alpha} \right).
\end{aligned} \tag{29}$$

For the symmetric parts of $P^{\alpha\beta}_{JK}$ and $Q^{\alpha\beta}_{J}$, we obtain

$$
\begin{aligned}
&\left(P^{(\alpha\beta)}_{JK} + \mathrm{i}g\,\tilde{M}_{LJ}Q^{(\alpha\beta)}_{L}\,\overline{\varphi}_K - \mathrm{i}g\,\varphi_J\,\overline{Q}^{(\alpha\beta)}_{L}\tilde{M}_{LK}\right)\\
&\cdot\left(\tfrac{\partial u_{KN}}{\partial x^\beta}a_{NI\alpha}\overline{u}_{IJ} + u_{KL}a_{LI\alpha}\tfrac{\partial\overline{u}_{IJ}}{\partial x^\beta} + \tfrac{1}{\mathrm{i}g}\tfrac{\partial u_{KI}}{\partial x^\alpha}\tfrac{\partial\overline{u}_{IJ}}{\partial x^\beta} + \tfrac{1}{\mathrm{i}g}\tfrac{\partial^2 u_{KI}}{\partial x^\alpha\partial x^\beta}\overline{u}_{IJ}\right)\\
&+\left(\tilde{M}_{LJ}Q^{(\alpha\beta)}_{L}\,\tfrac{\partial\overline{\varphi}_K}{\partial x^\beta} - \tfrac{\partial\varphi_J}{\partial x^\beta}\,\overline{Q}^{(\alpha\beta)}_{L}\tilde{M}_{LK}\right)\mathrm{i}g\,A_{KJ\alpha}\\
&+\overline{Q}^{(\alpha\beta)}_{L}\tilde{M}_{LK}\left(\tfrac{\partial u_{KI}}{\partial x^\beta}M_{IJ}b_{J\alpha} + \tfrac{\partial^2\varphi_K}{\partial x^\alpha\partial x^\beta}\right)\\
&+\left(\overline{b}_{J\alpha}M_{IJ}\tfrac{\partial\overline{u}_{IK}}{\partial x^\beta} + \tfrac{\partial^2\overline{\varphi}_K}{\partial x^\alpha\partial x^\beta}\right)\tilde{M}_{LK}Q^{(\alpha\beta)}_{L}\\
&=\left(P^{(\alpha\beta)}_{JK} + \mathrm{i}g\,\tilde{M}_{LJ}Q^{(\alpha\beta)}_{L}\,\overline{\varphi}_K - \mathrm{i}g\,\varphi_J\,\overline{Q}^{(\alpha\beta)}_{L}\tilde{M}_{LK}\right)\\
&\left(\tfrac{\partial A_{KJ\alpha}}{\partial x^\beta} - u_{KL}\tfrac{\partial a_{LI\alpha}}{\partial x^\beta}\overline{u}_{IJ}\right)\\
&+\overline{Q}^{(\alpha\beta)}_{L}\tilde{M}_{LK}\left(\tfrac{\partial u_{KI}}{\partial x^\beta}M_{IJ}b_{J\alpha} + \tfrac{\partial^2\varphi_K}{\partial x^\alpha\partial x^\beta} - \mathrm{i}g\,A_{KJ\alpha}\tfrac{\partial\varphi_J}{\partial x^\beta}\right)\\
&+\left(\overline{b}_{J\alpha}M_{IJ}\tfrac{\partial\overline{u}_{IK}}{\partial x^\beta} + \tfrac{\partial^2\overline{\varphi}_K}{\partial x^\alpha\partial x^\beta} + \mathrm{i}g\,\tfrac{\partial\overline{\varphi}_J}{\partial x^\beta}A_{JK\alpha}\right)\tilde{M}_{LK}Q^{(\alpha\beta)}_{L}\\
&=\tfrac{1}{2}P^{\alpha\beta}_{JK}\left(\tfrac{\partial A_{KJ\alpha}}{\partial x^\beta} + \tfrac{\partial A_{KJ\beta}}{\partial x^\alpha}\right) + \tfrac{1}{2}\overline{Q}^{\alpha\beta}_{K}\left(\tfrac{\partial B_{K\alpha}}{\partial x^\beta} + \tfrac{\partial B_{K\beta}}{\partial x^\alpha}\right)\\
&+\tfrac{1}{2}\left(\tfrac{\partial\overline{B}_{K\alpha}}{\partial x^\beta} + \tfrac{\partial\overline{B}_{K\beta}}{\partial x^\alpha}\right)Q^{\alpha\beta}_{K}\\
&-\tfrac{1}{2}p^{\alpha\beta}_{JK}\left(\tfrac{\partial a_{KJ\alpha}}{\partial x^\beta} + \tfrac{\partial a_{KJ\beta}}{\partial x^\alpha}\right)\\
&-\tfrac{1}{2}\overline{q}^{\alpha\beta}_{K}\left(\tfrac{\partial b_{K\alpha}}{\partial x^\beta} + \tfrac{\partial b_{K\beta}}{\partial x^\alpha}\right) - \tfrac{1}{2}\left(\tfrac{\partial\overline{b}_{K\alpha}}{\partial x^\beta} + \tfrac{\partial\overline{b}_{K\beta}}{\partial x^\alpha}\right)q^{\alpha\beta}_{K}.
\end{aligned}
\tag{30}
$$

In summary, by inserting the transformation rules into Eq. (27), the divergence of the explicitly x^μ-dependent terms of $\tilde{F}^\mu_2$ — and hence the difference of transformed and original Hamiltonians — can be expressed completely in terms of the canonical variables as

$$
\begin{aligned}
&\left.\tfrac{\partial\tilde{F}^\alpha_2}{\partial x^\alpha}\right|_{\mathrm{expl}} = \mathrm{i}g\left(\overline{\Pi}^\alpha_K\Phi_J - \overline{\Phi}_K\Pi^\alpha_J\right)A_{KJ\alpha}\\
&+\overline{\Pi}^\alpha_K M_{KJ}B_{J\alpha} + \overline{B}_{K\alpha}M_{JK}\Pi^\alpha_J\\
&-\mathrm{i}g\left(\overline{\pi}^\alpha_K\phi_J - \overline{\phi}_K\pi^\alpha_J\right)a_{KJ\alpha} - \left(\overline{\pi}^\alpha_K M_{KJ}b_{J\alpha} + \overline{b}_{K\alpha}M_{JK}\pi^\alpha_J\right)\\
&-\tfrac{1}{2}\mathrm{i}g\,P^{\alpha\beta}_{JK}\left(A_{KI\alpha}A_{IJ\beta} - A_{KI\beta}A_{IJ\alpha}\right)\\
&+\tfrac{1}{2}\mathrm{i}g\,p^{\alpha\beta}_{JK}\left(a_{KI\alpha}a_{IJ\beta} - a_{KI\beta}a_{IJ\alpha}\right)\\
&+\tfrac{1}{2}\mathrm{i}g\left(\overline{B}_{J\beta}M_{KJ}A_{KI\alpha}\tilde{M}_{IL} - \overline{B}_{J\alpha}M_{KJ}A_{KI\beta}\tilde{M}_{LI}\right)Q^{\alpha\beta}_{L}\\
&-\tfrac{1}{2}\mathrm{i}g\,\overline{Q}^{\alpha\beta}_{L}\left(\tilde{M}_{LI}A_{IK\alpha}M_{KJ}B_{J\beta} - \tilde{M}_{LI}A_{IK\beta}M_{KJ}B_{J\alpha}\right)\\
&-\tfrac{1}{2}\mathrm{i}g\left(\overline{b}_{J\beta}M_{KJ}a_{KI\alpha}\tilde{M}_{IL} - \overline{b}_{J\alpha}M_{KJ}a_{KI\beta}\tilde{M}_{LI}\right)q^{\alpha\beta}_{L}\\
&+\tfrac{1}{2}\mathrm{i}g\,\overline{q}^{\alpha\beta}_{L}\left(\tilde{M}_{LI}a_{IK\alpha}M_{KJ}b_{J\beta} - \tilde{M}_{LI}a_{IK\beta}M_{KJ}b_{J\alpha}\right)\\
&+\tfrac{1}{2}P^{\alpha\beta}_{JK}\left(\tfrac{\partial A_{KJ\alpha}}{\partial x^\beta} + \tfrac{\partial A_{KJ\beta}}{\partial x^\alpha}\right) + \tfrac{1}{2}\overline{Q}^{\alpha\beta}_{K}\left(\tfrac{\partial B_{K\alpha}}{\partial x^\beta} + \tfrac{\partial B_{K\beta}}{\partial x^\alpha}\right)\\
&+\tfrac{1}{2}\left(\tfrac{\partial\overline{B}_{K\alpha}}{\partial x^\beta} + \tfrac{\partial\overline{B}_{K\beta}}{\partial x^\alpha}\right)Q^{\alpha\beta}_{K}\\
&-\tfrac{1}{2}p^{\alpha\beta}_{JK}\left(\tfrac{\partial a_{KJ\alpha}}{\partial x^\beta} + \tfrac{\partial a_{KJ\beta}}{\partial x^\alpha}\right) - \tfrac{1}{2}\overline{q}^{\alpha\beta}_{K}\left(\tfrac{\partial b_{K\alpha}}{\partial x^\beta} + \tfrac{\partial b_{K\beta}}{\partial x^\alpha}\right)\\
&-\tfrac{1}{2}\left(\tfrac{\partial\overline{b}_{K\alpha}}{\partial x^\beta} + \tfrac{\partial\overline{b}_{K\beta}}{\partial x^\alpha}\right)q^{\alpha\beta}_{K}.
\end{aligned}
$$

We observe that *all* u_{IJ} and φ_I-dependencies of Eq. (27) were expressed *symmetrically* in terms of both the original and the transformed complex base fields ϕ_J, Φ_J and 4-vector gauge fields $\boldsymbol{a}_{JK}, \boldsymbol{A}_{JK}, \boldsymbol{b}_J, \boldsymbol{B}_J$, in conjunction

with their respective canonical momenta. Consequently, an amended Hamiltonian $\mathcal{H}_2$ of the form

$$\begin{aligned}
\mathcal{H}_2 &= \mathcal{H}(\pi,\phi,x) \\
&+\mathrm{i}g\left(\overline{\pi}^\alpha_K\phi_J - \overline{\phi}_K\pi^\alpha_J\right)a_{KJ\alpha} + \overline{\pi}^\alpha_K M_{KJ} b_{J\alpha} + \overline{b}_{K\alpha} M_{JK}\pi^\alpha_J \\
&-\tfrac{1}{2}\mathrm{i}g\,p^{\alpha\beta}_{JK}\left(a_{KI\alpha}\,a_{IJ\beta} - a_{KI\beta}\,a_{IJ\alpha}\right) + \tfrac{1}{2}p^{\alpha\beta}_{JK}\left(\tfrac{\partial a_{KJ\alpha}}{\partial x^\beta} + \tfrac{\partial a_{KJ\beta}}{\partial x^\alpha}\right) \\
&+\tfrac{1}{2}\mathrm{i}g\left(\overline{b}_{J\beta}M_{KJ}a_{KI\alpha} - \overline{b}_{J\alpha}M_{KJ}a_{KI\beta}\right)\tilde{M}_{LI}q^{\alpha\beta}_L \\
&-\tfrac{1}{2}\mathrm{i}g\,\overline{q}^{\alpha\beta}_L\tilde{M}_{LI}\left(a_{IK\alpha}M_{KJ}b_{J\beta} - a_{IK\beta}M_{KJ}b_{J\alpha}\right) \\
&+\tfrac{1}{2}\overline{q}^{\alpha\beta}_K\left(\tfrac{\partial b_{K\alpha}}{\partial x^\beta} + \tfrac{\partial b_{K\beta}}{\partial x^\alpha}\right) + \tfrac{1}{2}\left(\tfrac{\partial \overline{b}_{K\alpha}}{\partial x^\beta} + \tfrac{\partial \overline{b}_{K\beta}}{\partial x^\alpha}\right)q^{\alpha\beta}_K
\end{aligned} \tag{31}$$

is then transformed according to the general rule (11)

$$\mathcal{H}'_2 = \mathcal{H}_2 + \left.\frac{\partial\tilde{F}^\alpha_2}{\partial x^\alpha}\right|_{\mathrm{expl}}$$

into the new Hamiltonian

$$\begin{aligned}
\mathcal{H}'_2 &= \mathcal{H}(\mathbf{\Pi},\phi,x) \\
&+\mathrm{i}g\left(\overline{\Pi}^\alpha_K\Phi_J - \overline{\Phi}_K\Pi^\alpha_J\right)A_{KJ\alpha} + \overline{\Pi}^\alpha_K M_{KJ} B_{J\alpha} + \overline{B}_{K\alpha} M_{JK}\Pi^\alpha_J \\
&-\tfrac{1}{2}\mathrm{i}g\,P^{\alpha\beta}_{JK}\left(A_{KI\alpha}\,A_{IJ\beta} - A_{KI\beta}\,A_{IJ\alpha}\right) + \tfrac{1}{2}P^{\alpha\beta}_{JK}\left(\tfrac{\partial A_{KJ\alpha}}{\partial x^\beta} + \tfrac{\partial A_{KJ\beta}}{\partial x^\alpha}\right) \\
&+\tfrac{1}{2}\mathrm{i}g\left(\overline{B}_{J\beta}M_{KJ}A_{KI\alpha} - \overline{B}_{J\alpha}M_{KJ}A_{KI\beta}\right)\tilde{M}_{LI}Q^{\alpha\beta}_L \\
&-\tfrac{1}{2}\mathrm{i}g\,\overline{Q}^{\alpha\beta}_L\tilde{M}_{LI}\left(A_{IK\alpha}M_{KJ}B_{J\beta} - A_{IK\beta}M_{KJ}B_{J\alpha}\right) \\
&+\tfrac{1}{2}\overline{Q}^{\alpha\beta}_K\left(\tfrac{\partial B_{K\alpha}}{\partial x^\beta} + \tfrac{\partial B_{K\beta}}{\partial x^\alpha}\right) + \tfrac{1}{2}\left(\tfrac{\partial \overline{B}_{K\alpha}}{\partial x^\beta} + \tfrac{\partial \overline{B}_{K\beta}}{\partial x^\alpha}\right)Q^{\alpha\beta}_K.
\end{aligned} \tag{32}$$

The entire transformation is thus *form-conserving* provided that the original Hamiltonian $\mathcal{H}(\pi,\phi,x)$ is also form-invariant if expressed in terms of the new fields, $\mathcal{H}(\mathbf{\Pi},\phi,x) = \mathcal{H}(\pi,\phi,x)$, according to the transformation rules (14). In other words, $\mathcal{H}(\pi,\phi,x)$ must be form-invariant under the corresponding *global* gauge transformation.

As a common feature of all gauge transformation theories, we must ensure that the transformation rules for the gauge fields and their conjugates are consistent with the *field equations* for the gauge fields that follow from final form-invariant amended Hamiltonians, $\mathcal{H}_3 = \mathcal{H}_2 + \mathcal{H}_{\mathrm{kin}}$ and $\mathcal{H}'_3 = \mathcal{H}'_2 + \mathcal{H}'_{\mathrm{kin}}$. In other words, $\mathcal{H}_{\mathrm{kin}}$ and the form-alike $\mathcal{H}'_{\mathrm{kin}}$ must be chosen in a way that the transformation properties of the canonical equations for the gauge fields emerging from $\mathcal{H}_3$ and $\mathcal{H}'_3$ are compatible with the canonical transformation rules (19). These requirements *uniquely determine* the form of both $\mathcal{H}_{\mathrm{kin}}$ and $\mathcal{H}'_{\mathrm{kin}}$. Thus, the Hamiltonians (31) and (32) must be further amended by "kinetic" terms $\mathcal{H}_{\mathrm{kin}}$ and $\mathcal{H}'_{\mathrm{kin}}$ that describe the dynamics of the free 4-vector gauge fields, $\boldsymbol{a}_{KJ}, \boldsymbol{b}_J$ and $\boldsymbol{A}_{KJ}, \boldsymbol{B}_J$, respectively. Of course, $\mathcal{H}_{\mathrm{kin}}$ must be form-invariant as well if expressed in the transformed dynamical variables in order to ensure the overall form-invariance of the final Hamiltonian. An expression that fulfills this requirement is obtained from Eqs. (22) and (23)

$$\begin{aligned}
\mathcal{H}_{\mathrm{kin}} \;&= -\tfrac{1}{2}\overline{q}^{\alpha\beta}_J\,q_{J\alpha\beta} - \tfrac{1}{4}\left(p^{\alpha\beta}_{IJ} + \mathrm{i}g\,\tilde{M}_{LI}q^{\alpha\beta}_L\,\overline{\phi}_J - \mathrm{i}g\,\phi_I\,\overline{q}^{\alpha\beta}_L\tilde{M}_{LJ}\right) \\
&\quad\cdot\left(p_{JI\alpha\beta} + \mathrm{i}g\,\tilde{M}_{KJ}q_{K\alpha\beta}\,\overline{\phi}_I - \mathrm{i}g\,\phi_J\,\overline{q}_{K\alpha\beta}\tilde{M}_{KI}\right).
\end{aligned} \tag{33}$$

The condition for the first term to be form-invariant is

$$\begin{aligned}
\overline{q}^{\alpha\beta}_J\,q_{J\alpha\beta} \;&= \overline{Q}^{\alpha\beta}_L\,\tilde{M}_{LK}\,u_{KI}\;\underbrace{M_{IJ}\,M_{NJ}}_{\stackrel{!}{=}\delta_{IN}(\det M)^2}\;\overline{u}_{NR}\,\tilde{M}_{SR}\,Q_{S\alpha\beta} \\
&= (\det M)^2\,\overline{Q}^{\alpha\beta}_L\;\underbrace{\tilde{M}_{LK}\,\tilde{M}_{JK}}_{\stackrel{!}{=}\delta_{LJ}(\det M)^{-2}}\;Q_{J\alpha\beta} \\
&= \overline{Q}^{\alpha\beta}_J\,Q_{J\alpha\beta}
\end{aligned}$$

The mass matrix M must thus be orthogonal

$$M\,M^T = \mathbb{1}\,(\det M)^2. \tag{34}$$

From $\mathcal{H}_3$ and, correspondingly, from $\mathcal{H}_3'$, we will work out the condition for the canonical field equations to be consistent with the canonical transformation rules (19) for the gauge fields and their conjugates (22).

With $\mathcal{H}_{\rm kin}$ from Eq. (33), the total amended Hamiltonian $\mathcal{H}_3$ is now given by

(35)

$$\begin{gathered}
\mathcal{H}_3 = \mathcal{H}_2 + \mathcal{H}_{\rm kin} = \mathcal{H} + \mathcal{H}_{\rm g} \\
\mathcal{H}_{\rm g} = {\rm i}g\left(\overline{\pi}^\alpha_K \phi_J - \overline{\phi}_K \pi^\alpha_J\right) a_{KJ\alpha} - \tfrac{1}{2}{\rm i}g\, p^{\alpha\beta}_{KJ}\left(a_{JI\alpha}\, a_{IK\beta} - a_{JI\beta}\, a_{IK\alpha}\right) \\
+\tfrac{1}{2} p^{\alpha\beta}_{KJ}\left(\tfrac{\partial a_{JK\alpha}}{\partial x^\beta} + \tfrac{\partial a_{JK\beta}}{\partial x^\alpha}\right) + \tfrac{1}{2}\overline{q}^{\alpha\beta}_J\left(\tfrac{\partial b_{J\alpha}}{\partial x^\beta} + \tfrac{\partial b_{J\beta}}{\partial x^\alpha}\right) \\
+\tfrac{1}{2}\left(\tfrac{\partial \overline{b}_{J\alpha}}{\partial x^\beta} + \tfrac{\partial \overline{b}_{J\beta}}{\partial x^\alpha}\right) q^{\alpha\beta}_J \\
+\overline{\pi}^\alpha_K M_{KJ} b_{J\alpha} + \overline{b}_{K\alpha} M_{JK} \pi^\alpha_J \\
+\tfrac{1}{2}{\rm i}g\left(\overline{b}_{J\beta} M_{KJ} a_{KI\alpha} - \overline{b}_{J\alpha} M_{KJ} a_{KI\beta}\right) \tilde{M}_{LI} q^{\alpha\beta}_L \\
-\tfrac{1}{2}{\rm i}g\,\overline{q}^{\alpha\beta}_L \tilde{M}_{LI}\left(a_{IK\alpha} M_{KJ} b_{J\beta} - a_{IK\beta} M_{KJ} b_{J\alpha}\right) - \tfrac{1}{2}\overline{q}^{\alpha\beta}_J\, q_{J\alpha\beta} \\
-\tfrac{1}{4}\Big(p^{\alpha\beta}_{IJ} + {\rm i}g\,\tilde{M}_{LI} q^{\alpha\beta}_L\,\overline{\phi}_J - {\rm i}g\,\phi_I\,\overline{q}^{\alpha\beta}_L \tilde{M}_{LJ}\Big) \\
\Big(p_{JI\alpha\beta} + {\rm i}g\,\tilde{M}_{KJ} q_{K\alpha\beta}\,\overline{\phi}_I - {\rm i}g\,\phi_J\,\overline{q}_{K\alpha\beta}\tilde{M}_{KI}\Big).
\end{gathered}$$

In the Hamiltonian description, the partial derivatives of the fields in (35) do *not* constitute canonical variables and must hence be regarded as x^μ-dependent coefficients when setting up the canonical field equations. The relation of the canonical momenta $p^{\mu\nu}_{NM}$ to the derivatives of the fields, $\partial a_{MN\mu}/\partial x^\nu$, is generally provided by the first canonical field equation (5). This means for the particular Hamiltonian (35)

$$\begin{gathered}
\tfrac{\partial a_{MN\mu}}{\partial x^\nu} = \tfrac{\partial \mathcal{H}_{\rm g}}{\partial p^{\mu\nu}_{NM}} \\
= -\tfrac{1}{2}{\rm i}g\left(a_{MI\mu}\, a_{IN\nu} - a_{MI\nu}\, a_{IN\mu}\right) + \tfrac{1}{2}\left(\tfrac{\partial a_{MN\mu}}{\partial x^\nu} + \tfrac{\partial a_{MN\nu}}{\partial x^\mu}\right) \\
-\tfrac{1}{2} p_{MN\mu\nu} - \tfrac{1}{2}{\rm i}g\left(\tilde{M}_{IM} q_{I\mu\nu}\,\overline{\phi}_N - \phi_M\,\overline{q}_{I\mu\nu}\tilde{M}_{IN}\right),
\end{gathered}$$

hence

$$\begin{gathered}
p_{KJ\mu\nu} = \tfrac{\partial a_{KJ\nu}}{\partial x^\mu} - \tfrac{\partial a_{KJ\mu}}{\partial x^\nu} \\
+{\rm i}g\left(a_{KI\nu}\, a_{IJ\mu} - a_{KI\mu}\, a_{IJ\nu} - \tilde{M}_{IK} q_{I\mu\nu}\,\overline{\phi}_J + \phi_K\,\overline{q}_{I\mu\nu}\tilde{M}_{IJ}\right).
\end{gathered} \tag{36}$$

Rewriting Eq. (36) in the form

$$\begin{gathered}
p_{KJ\mu\nu} + {\rm i}g\tilde{M}_{IK} q_{I\mu\nu}\overline{\phi}_J - {\rm i}g\phi_K \overline{q}_{I\mu\nu}\tilde{M}_{IJ} \\
= \tfrac{\partial a_{KJ\nu}}{\partial x^\mu} - \tfrac{\partial a_{KJ\mu}}{\partial x^\nu} + {\rm i}g\left(a_{KI\nu} a_{IJ\mu} - a_{KI\mu} a_{IJ\nu}\right) \\
= f_{KJ\mu\nu},
\end{gathered}$$

we realize that the left-hand side transforms homogeneously according to Eq. (23). From Eq. (26), we already know that the same rule applies for the $\boldsymbol{f}_{\mu\nu}$. The canonical equation (36) is thus generally consistent with the canonical transformation rules.

The corresponding reasoning applies for the canonical momenta $q_{J\mu\nu}$ and $\overline{q}_{J\mu\nu}$

$$\begin{aligned}
\frac{\partial b_{N\mu}}{\partial x^\nu} &= \frac{\partial \mathcal{H}_{\mathrm{g}}}{\partial \overline{q}_N^{\mu\nu}} = \\
-\tfrac{1}{2} q_{N\mu\nu} - \tfrac{1}{2}\mathrm{i}g\,\tilde{M}_{NI}\,(a_{IK\mu}M_{KJ}\,b_{J\nu} - a_{IK\nu}M_{KJ}\,b_{J\mu}) & \\
+\tfrac{1}{2}\left(\tfrac{\partial b_{N\mu}}{\partial x^\nu} + \tfrac{\partial b_{N\nu}}{\partial x^\mu}\right) & \\
+\tfrac{1}{2}\mathrm{i}g\,\tilde{M}_{NI}\left(p_{IJ\mu\nu} + \mathrm{i}g\,\tilde{M}_{KI}q_{K\mu\nu}\,\overline{\phi}_J - \mathrm{i}g\,\phi_I\,\overline{q}_{K\mu\nu}\tilde{M}_{KJ}\right)\phi_J & \\
\frac{\partial \overline{b}_{N\mu}}{\partial x^\nu} &= \frac{\partial \mathcal{H}_{\mathrm{g}}}{\partial q_N^{\mu\nu}} = \\
-\tfrac{1}{2}\overline{q}_{N\mu\nu} + \tfrac{1}{2}\mathrm{i}g\left(\overline{b}_{J\nu}M_{KJ}\,a_{KI\mu} - \overline{b}_{J\mu}M_{KJ}\,a_{KI\nu}\right)\tilde{M}_{NI} & \\
+\tfrac{1}{2}\left(\tfrac{\partial \overline{b}_{N\mu}}{\partial x^\nu} + \tfrac{\partial \overline{b}_{N\nu}}{\partial x^\mu}\right) & \\
-\tfrac{1}{2}\mathrm{i}g\,\overline{\phi}_J\left(p_{JI\mu\nu} + \mathrm{i}g\,\tilde{M}_{KJ}q_{K\mu\nu}\,\overline{\phi}_I - \mathrm{i}g\,\phi_J\,\overline{q}_{K\mu\nu}\tilde{M}_{KI}\right)\tilde{M}_{NI}, &
\end{aligned}$$

hence with the canonical equation (36)

$$\begin{aligned}
q_{J\mu\nu} &= \tfrac{\partial b_{J\nu}}{\partial x^\mu} - \tfrac{\partial b_{J\mu}}{\partial x^\nu} + \mathrm{i}g\,\tilde{M}_{JI}\,(a_{IK\nu}M_{KL}\,b_{L\mu} - a_{IK\mu}M_{KL}\,b_{L\nu}) \\
&\quad + \mathrm{i}g\,\tilde{M}_{JI}\left(\tfrac{\partial a_{IK\nu}}{\partial x^\mu} - \tfrac{\partial a_{IK\mu}}{\partial x^\nu} + \mathrm{i}g\,(a_{IL\nu}\,a_{LK\mu} - a_{IL\mu}\,a_{LK\nu})\right)\phi_K \\
\overline{q}_{J\mu\nu} &= \tfrac{\partial \overline{b}_{J\nu}}{\partial x^\mu} - \tfrac{\partial \overline{b}_{J\mu}}{\partial x^\nu} - \mathrm{i}g\left(\overline{b}_{L\mu}M_{KL}\,a_{KI\nu} - \overline{b}_{L\nu}M_{KL}\,a_{KI\mu}\right)\tilde{M}_{JI} \\
&\quad - \mathrm{i}g\,\overline{\phi}_K\left(\tfrac{\partial a_{KI\nu}}{\partial x^\mu} - \tfrac{\partial a_{KI\mu}}{\partial x^\nu} + \mathrm{i}g\,(a_{KL\nu}\,a_{LI\mu} - a_{KL\mu}\,a_{LI\nu})\right)\tilde{M}_{JI}.
\end{aligned} \tag{37}$$

In order to check whether these canonical equations—which are complex conjugate to each other—are also compatible with the canonical transformation rules, we rewrite the first one concisely in matrix notation for the transformed fields

$$\begin{aligned}
M\boldsymbol{Q}_{\mu\nu} &= \tfrac{\partial M\boldsymbol{B}_\nu}{\partial x^\mu} - \tfrac{\partial M\boldsymbol{B}_\mu}{\partial x^\nu} + \mathrm{i}g\,(\boldsymbol{A}_\nu M\,\boldsymbol{B}_\mu - \boldsymbol{A}_\mu M\,\boldsymbol{B}_\nu) \\
&\quad + \mathrm{i}g\left(\tfrac{\partial \boldsymbol{A}_\nu}{\partial x^\mu} - \tfrac{\partial \boldsymbol{A}_\mu}{\partial x^\nu} + \mathrm{i}g\,(\boldsymbol{A}_\nu\boldsymbol{A}_\mu - \boldsymbol{A}_\mu\boldsymbol{A}_\nu)\right)\phi.
\end{aligned}$$

Applying now the transformation rules for the gauge fields $\boldsymbol{A}_\nu, \boldsymbol{B}_\mu$ from Eqs. (20), and for the base fields ϕ from Eqs. (12), we find

$$\begin{aligned}
M\boldsymbol{Q}_{\mu\nu} &= U\left[\tfrac{\partial M\boldsymbol{b}_\nu}{\partial x^\mu} - \tfrac{\partial M\boldsymbol{b}_\mu}{\partial x^\nu} + \mathrm{i}g\,(\boldsymbol{a}_\nu M\,\boldsymbol{b}_\mu - \boldsymbol{a}_\mu M\,\boldsymbol{b}_\nu)\right. \\
&\quad \left. + \mathrm{i}g\left(\tfrac{\partial \boldsymbol{a}_\nu}{\partial x^\mu} - \tfrac{\partial \boldsymbol{a}_\mu}{\partial x^\nu} + \mathrm{i}g\,(\boldsymbol{a}_\nu\boldsymbol{a}_\mu - \boldsymbol{a}_\mu\boldsymbol{a}_\nu)\right)\phi\right] \\
&= UM\,\boldsymbol{q}_{\mu\nu}.
\end{aligned}$$

The canonical equations (37) are thus compatible with the canonical transformation rules (26) provided that

$$\tilde{M}^T = \frac{M}{(\det M)^2}.$$

Thus, the mass matrix M must be *orthogonal*. This restriction was already encountered with Eq. (34).

We observe that both $p_{KJ\mu\nu}$ and $q_{J\mu\nu}, \overline{q}_{J\mu\nu}$ occur to be skew-symmetric in the indices μ, ν. Here, this feature emerges from the canonical formalism and does not have to be postulated. Consequently, all products with the momenta in the Hamiltonian (35) that are *symmetric* in μ, ν must vanish. As these terms only contribute to the first canonical equations, we may omit them from $\mathcal{H}_{\mathrm{g}}$ if we simultaneously *define* $p_{JK\mu\nu}$ and $q_{J\mu\nu}$ to be skew-symmetric in μ, ν. With regard to the ensuing canonical equations, the gauge Hamiltonian $\mathcal{H}_{\mathrm{g}}$ from

Eq. (35) is then equivalent to

$$\begin{aligned}\mathcal{H}_{\mathrm{g}} = {} & \mathrm{i}g\left(\overline{\pi}_K^{\beta}\phi_J - \overline{\phi}_K\pi_J^{\beta}\right)a_{KJ\beta} - \mathrm{i}g\,p_{JI}^{\alpha\beta}a_{IK\alpha}\,a_{KJ\beta} - \tfrac{1}{2}\,\overline{q}_J^{\alpha\beta}\,q_{J\alpha\beta} \\ & + \left(\overline{\pi}_K^{\beta} - \mathrm{i}g\,\overline{q}_L^{\alpha\beta}\tilde{M}_{LI}a_{IK\alpha}\right)M_{KJ}b_{J\beta} \\ & + \overline{b}_{K\beta}M_{JK}\left(\pi_J^{\beta} + \mathrm{i}g\,a_{JI\alpha}\tilde{M}_{LI}q_L^{\alpha\beta}\right) \\ & - \tfrac{1}{4}\Big(p_{IJ}^{\alpha\beta} + \mathrm{i}g\,\tilde{M}_{LI}q_L^{\alpha\beta}\,\overline{\phi}_J - \mathrm{i}g\,\phi_I\,\overline{q}_L^{\alpha\beta}\tilde{M}_{LJ}\Big) \\ & \Big(p_{JI\alpha\beta} + \mathrm{i}g\,\tilde{M}_{KJ}q_{K\alpha\beta}\,\overline{\phi}_I - \mathrm{i}g\,\phi_J\,\overline{q}_{K\alpha\beta}\tilde{M}_{KI}\Big) \\ & p_{JK}^{\mu\nu} \stackrel{!}{=} -p_{JK}^{\nu\mu}, \qquad q_J^{\mu\nu} \stackrel{!}{=} -q_J^{\nu\mu}. \end{aligned} \tag{38}$$

Setting the mass matrix M to zero, $\mathcal{H}_{\mathrm{g}}$ reduces to the gauge Hamiltonian of the homogeneous U(N) gauge theory [11]. The other terms describe the dynamics of the 4-vector gauge fields $\boldsymbol{b}_J$. From the locally gauge-invariant Hamiltonian (35), the canonical equations for the base fields $\phi_I, \overline{\phi}_I$ are given by

$$\begin{aligned}\left.\frac{\partial\phi_I}{\partial x^{\mu}}\right|_{\mathcal{H}_3} &= \frac{\partial\mathcal{H}_3}{\partial\overline{\pi}_I^{\mu}} &&= \tfrac{\partial\mathcal{H}}{\partial\overline{\pi}_I^{\mu}} + \mathrm{i}g\,a_{IJ\mu}\phi_J + M_{IJ}b_{J\mu} \\ \left.\frac{\partial\overline{\phi}_I}{\partial x^{\mu}}\right|_{\mathcal{H}_3} &= \frac{\partial\mathcal{H}_3}{\partial\pi_I^{\mu}} &&= \tfrac{\partial\mathcal{H}}{\partial\pi_I^{\mu}} - \mathrm{i}g\,\overline{\phi}_J a_{JI\mu} + \overline{b}_{J\mu}M_{IJ}. \end{aligned} \tag{39}$$

These equations represent the generalized "minimum coupling rules" for our particular case of a system of two sets of gauge fields, $\boldsymbol{a}_{JK}$ and $\boldsymbol{b}_J$.

The canonical field equation from the $\boldsymbol{b}_J, \overline{\boldsymbol{b}}_J$ dependencies of $\mathcal{H}_{\mathrm{g}}$ follow as

$$\begin{aligned}\frac{\partial q_K^{\mu\alpha}}{\partial x^{\alpha}} &= -\tfrac{\partial\mathcal{H}_{\mathrm{g}}}{\partial\overline{b}_{K\mu}} = -M_{JK}\left(\pi_J^{\mu} + \mathrm{i}g\,a_{JI\alpha}\tilde{M}_{LI}q_L^{\alpha\mu}\right) \\ \frac{\partial\overline{q}_J^{\mu\alpha}}{\partial x^{\alpha}} &= -\tfrac{\partial\mathcal{H}_{\mathrm{g}}}{\partial b_{J\mu}} = \left(-\overline{\pi}_K^{\mu} + \mathrm{i}g\,\overline{q}_L^{\alpha\mu}\tilde{M}_{LI}a_{IK\alpha}\right)M_{KJ}. \end{aligned}$$

Inserting $\pi_J^{\alpha}, \overline{\pi}_J^{\alpha}$ as obtained from Eqs. (39) for a particular system Hamiltonian $\mathcal{H}$, terms proportional to b_I^{α} and $\overline{b}_I^{\alpha}$ emerge with no other dynamical variables involved. Such terms describe the masses of particles that are associated with the gauge fields $\boldsymbol{b}_I$.

Gauge-invariant Lagrangian

As the system Hamiltonian $\mathcal{H}$ does not depend on the gauge fields $\boldsymbol{a}_{KJ}$ and $\boldsymbol{b}_J$, the gauge Lagrangian $\mathcal{L}_{\mathrm{g}}$ that is equivalent to the gauge Hamiltonian $\mathcal{H}_{\mathrm{g}}$ from Eq. (35) is derived by means of the Legendre transformation

$$\mathcal{L}_{\mathrm{g}} = p_{JK}^{\alpha\beta}\frac{\partial a_{KJ\alpha}}{\partial x^{\beta}} + \overline{q}_J^{\alpha\beta}\frac{\partial b_{J\alpha}}{\partial x^{\beta}} + \frac{\partial\overline{b}_{J\alpha}}{\partial x^{\beta}}q_J^{\alpha\beta} - \mathcal{H}_{\mathrm{g}},$$

with $p_{JK}^{\mu\nu}$ from Eq. (36) and $q_J^{\mu\nu}, \overline{q}_J^{\mu\nu}$ from Eqs. (37). We thus have

$$\begin{aligned} p_{JK}^{\alpha\beta}\tfrac{\partial a_{KJ\alpha}}{\partial x^{\beta}} &= \tfrac{1}{2}p_{JK}^{\alpha\beta}\left(\tfrac{\partial a_{KJ\alpha}}{\partial x^{\beta}} - \tfrac{\partial a_{KJ\beta}}{\partial x^{\alpha}}\right) + \tfrac{1}{2}p_{JK}^{\alpha\beta}\left(\tfrac{\partial a_{KJ\alpha}}{\partial x^{\beta}} + \tfrac{\partial a_{KJ\beta}}{\partial x^{\alpha}}\right) \\ &= -\tfrac{1}{2}p_{JK}^{\alpha\beta}\,p_{KJ\alpha\beta} + \tfrac{1}{2}p_{JK}^{\alpha\beta}\left(\tfrac{\partial a_{KJ\alpha}}{\partial x^{\beta}} + \tfrac{\partial a_{KJ\beta}}{\partial x^{\alpha}}\right) \\ & - \tfrac{1}{2}\mathrm{i}g\,p_{JK}^{\alpha\beta}\left(a_{KI\alpha}\,a_{IJ\beta} - a_{KI\beta}\,a_{IJ\alpha} - \tilde{M}_{IK}q_{I\beta\alpha}\,\overline{\phi}_J + \phi_K\,\overline{q}_{I\beta\alpha}\tilde{M}_{IJ}\right), \end{aligned}$$

and, similarly

$$\begin{gathered}
\overline{q}_J^{\,\alpha\beta}\,\tfrac{\partial b_{J\alpha}}{\partial x^\beta} = \\
-\tfrac{1}{2}\,\overline{q}_J^{\,\alpha\beta}\,q_{J\alpha\beta} - \tfrac{1}{2}\mathrm{i}g\,\overline{q}_J^{\,\alpha\beta}\tilde{M}_{JI}\,(a_{IK\alpha}M_{KL}\,b_{L\beta} - a_{IK\beta}M_{KL}\,b_{L\alpha}) \\
+\tfrac{1}{2}\mathrm{i}g\,\overline{q}_J^{\,\alpha\beta}\tilde{M}_{JI}\left(p_{IL\alpha\beta} + \mathrm{i}g\,\tilde{M}_{KI}q_{K\alpha\beta}\,\overline{\phi}_L - \mathrm{i}g\,\phi_I\,\overline{q}_{K\alpha\beta}\tilde{M}_{KL}\right)\phi_L \\
+\tfrac{1}{2}\overline{q}_J^{\,\alpha\beta}\left(\tfrac{\partial b_{J\alpha}}{\partial x^\beta} + \tfrac{\partial b_{J\beta}}{\partial x^\alpha}\right) \\
\tfrac{\partial \overline{b}_{J\alpha}}{\partial x^\beta}q_J^{\alpha\beta} = \\
-\tfrac{1}{2}\,\overline{q}_J^{\,\alpha\beta}\,q_{J\alpha\beta} + \tfrac{1}{2}\mathrm{i}g\,\left(\overline{b}_{L\beta}M_{KL}\,a_{KI\alpha} - \overline{b}_{L\alpha}M_{KL}\,a_{KI\beta}\right)\tilde{M}_{JI}q_J^{\alpha\beta} \\
-\tfrac{1}{2}\mathrm{i}g\,\overline{\phi}_I\left(p_{IL\alpha\beta} + \mathrm{i}g\,\tilde{M}_{KI}q_{K\alpha\beta}\,\overline{\phi}_L - \mathrm{i}g\,\phi_I\,\overline{q}_{K\alpha\beta}\tilde{M}_{KL}\right)\tilde{M}_{JL}q_J^{\alpha\beta} \\
+\tfrac{1}{2}\left(\tfrac{\partial \overline{b}_{J\alpha}}{\partial x^\beta} + \tfrac{\partial \overline{b}_{J\beta}}{\partial x^\alpha}\right)q_J^{\alpha\beta}.
\end{gathered}$$

With the gauge Hamiltonian $\mathcal{H}_\mathrm{g}$ from Eq. (35), the gauge Lagrangian $\mathcal{L}_\mathrm{g}$ is then

$$\begin{gathered}
\mathcal{L}_\mathrm{g} = -\tfrac{1}{2}\,\overline{q}_J^{\,\alpha\beta}q_{J\alpha\beta} \\
-\overline{\pi}_K^\alpha\,(\mathrm{i}g\,a_{KJ\alpha}\phi_J + M_{KJ}b_{J\alpha}) + \left(\mathrm{i}g\,\overline{\phi}_K a_{KJ\alpha} - \overline{b}_{K\alpha}M_{JK}\right)\pi_J^\alpha \\
-\tfrac{1}{4}\left(p_{IJ}^{\alpha\beta} + \mathrm{i}g\,\tilde{M}_{LI}q_L^{\alpha\beta}\,\overline{\phi}_J - \mathrm{i}g\,\phi_I\,\overline{q}_L^{\,\alpha\beta}\tilde{M}_{LJ}\right) \\
\cdot\left(p_{JI\alpha\beta} + \mathrm{i}g\,\tilde{M}_{KJ}q_{K\alpha\beta}\,\overline{\phi}_I - \mathrm{i}g\,\phi_J\,\overline{q}_{K\alpha\beta}\tilde{M}_{KI}\right)
\end{gathered}$$

According to Eq. (24) and the relation for the canonical momenta $p_{JI\alpha\beta}$ from Eq. (36), the last product can be rewritten as $-\frac{1}{4}f_{IJ}^{\alpha\beta}\,f_{JI\alpha\beta}$, thus

$$\begin{aligned}
\mathcal{L}_\mathrm{g} = -\tfrac{1}{4}f_{IJ}^{\alpha\beta}\,f_{JI\alpha\beta} - \tfrac{1}{2}\,\overline{q}_J^{\,\alpha\beta}q_{J\alpha\beta} - \overline{\pi}_K^\alpha\,(\mathrm{i}g\,a_{KJ\alpha}\phi_J + M_{KJ}b_{J\alpha}) + \\
\left(\mathrm{i}g\,\overline{\phi}_K a_{KJ\alpha} - \overline{b}_{K\alpha}M_{JK}\right)\pi_J^\alpha.
\end{aligned}$$

With regard to canonical variables $\overline{\pi}_K, \pi_K$, $\mathcal{L}_\mathrm{g}$ is still a Hamiltonian. The final total gauge-invariant Lagrangian $\mathcal{L}_3$ for the given system Hamiltonian $\mathcal{H}$ then emerges from the Legendre transformation

$$\begin{aligned}
\mathcal{L}_3 &= \mathcal{L}_\mathrm{g} + \overline{\pi}_J^\alpha\tfrac{\partial\phi_J}{\partial x^\alpha} + \tfrac{\partial\overline{\phi}_J}{\partial x^\alpha}\pi_J^\alpha - \mathcal{H}(\overline{\phi}_I, \phi_I, \overline{\pi}_I, \pi_I, x) \\
&= \overline{\pi}_J^\alpha\left(\tfrac{\partial\phi_J}{\partial x^\alpha} - \mathrm{i}g\,a_{JK\alpha}\phi_K - M_{JK}\,b_{K\alpha}\right) \\
&\quad + \left(\tfrac{\partial\overline{\phi}_J}{\partial x^\alpha} + \mathrm{i}g\,\overline{\phi}_K a_{KJ\alpha} - \overline{b}_{K\alpha}M_{JK}\right)\pi_J^\alpha \\
&\quad - \tfrac{1}{4}f_{IJ}^{\alpha\beta}\,f_{JI\alpha\beta} - \tfrac{1}{2}\,\overline{q}_J^{\,\alpha\beta}q_{J\alpha\beta} - \mathcal{H}(\,\overline{\phi}_I, \phi_I, \overline{\pi}_I, \pi_I, x).
\end{aligned} \tag{40}$$

As implied by the Lagrangian formalism, the dynamical variables are given by both the fields, ϕ_I, $\overline{\phi}_I$, $\boldsymbol{a}_{KJ}$, $\boldsymbol{b}_J$, and $\overline{\boldsymbol{b}}_J$, and their respective partial derivatives with respect to the independent variables, x^μ. Therefore, the momenta $\boldsymbol{q}_J$ and $\overline{\boldsymbol{q}}_J$ of the Hamiltonian description are no longer dynamical variables in $\mathcal{L}_\mathrm{g}$ but merely *abbreviations* for combinations of the Lagrangian dynamical variables, which are here given by Eqs. (37). The correlation of the momenta $\pi_I, \overline{\pi}_I$ of the base fields $\phi_I, \overline{\phi}_I$ to their derivatives are derived from the system Hamiltonian $\mathcal{H}$ via

$$\begin{aligned}
\frac{\partial\phi_I}{\partial x^\mu} &= \tfrac{\partial\mathcal{H}}{\partial\overline{\pi}_I^\mu} + \mathrm{i}g\,a_{IJ\mu}\phi_J + M_{IJ}b_{J\mu} \\
\frac{\partial\overline{\phi}_I}{\partial x^\mu} &= \tfrac{\partial\mathcal{H}}{\partial\pi_I^\mu} - \mathrm{i}g\,\overline{\phi}_J\,a_{JI\mu} + \overline{b}_{J\mu}M_{IJ},
\end{aligned} \tag{41}$$

which represents the "minimal coupling rule" for our particular system. Thus, for any *globally* gauge-invariant Hamiltonian $\mathcal{H}(\phi_I, \pi_I, x)$, the amended Lagrangian (40) with Eqs. (41) describes in the Lagrangian formalism the associated physical system that is invariant under *local* gauge transformations.

Klein-Gordon system Hamiltonian

As an example, we consider the generalized Klein-Gordon Hamiltonian [11] that describes an N-tuple of *massless* spin-0 fields

$$\mathcal{H}_{\mathrm{KG}} = \overline{\pi}_I^{\,\alpha}\, \pi_{I\alpha}.$$

This Hamiltonian is clearly invariant under the inhomogeneous global gauge transformation (14). The reason for defining a *massless* system Hamiltonian $\mathcal{H}$ is that a mass term of the form $\overline{\phi}_I M_{JI} M_{JK} \phi_K$ that is contained in the general Klein-Gordon Hamiltonian is *not invariant* under the inhomogeneous gauge transformation from Eq. (14). According to Eqs. (40) and (41), the corresponding locally gauge-invariant Lagrangian $\mathcal{L}_{3,\mathrm{KG}}$ is then

$$\boxed{\mathcal{L}_{3,\mathrm{KG}} = \overline{\pi}_I^{\,\alpha}\, \pi_{I\alpha} - \tfrac{1}{4} f_{JK}^{\alpha\beta}\, f_{KJ\alpha\beta} - \tfrac{1}{2} \overline{q}_J^{\,\alpha\beta} q_{J\alpha\beta},} \tag{42}$$

with

$$\begin{aligned}
f_{KJ\mu\nu} &= \tfrac{\partial a_{KJ\nu}}{\partial x^\mu} - \tfrac{\partial a_{KJ\mu}}{\partial x^\nu} + \mathrm{i}g\,(a_{KI\nu}\, a_{IJ\mu} - a_{KI\mu}\, a_{IJ\nu}) \\
q_{J\mu\nu} &= \tfrac{\partial b_{J\nu}}{\partial x^\mu} - \tfrac{\partial b_{J\mu}}{\partial x^\nu} \\
&+\mathrm{i}g\,\tilde{M}_{JI}\left(a_{IK\nu}\, M_{KL}\, b_{L\mu} - a_{IK\mu} M_{KL}\, b_{L\nu} + f_{IK\mu\nu}\,\phi_K\right) \\
\overline{q}_{J\mu\nu} &= \tfrac{\partial \overline{b}_{J\nu}}{\partial x^\mu} - \tfrac{\partial \overline{b}_{J\mu}}{\partial x^\nu} \\
&-\mathrm{i}g\left(\overline{b}_{L\mu}\, M_{KL}\, a_{KI\nu} - \overline{b}_{L\nu}\, M_{KL}\, a_{KI\mu} + \overline{\phi}_K\, f_{KI\mu\nu}\right)\tilde{M}_{JI} \\
\pi_{I\mu} &= \tfrac{\partial \phi_I}{\partial x^\mu} - \mathrm{i}g\, a_{IJ\mu}\phi_J - M_{IJ}\, b_{J\mu} \\
\overline{\pi}_{I\mu} &= \tfrac{\partial \overline{\phi}_I}{\partial x^\mu} + \mathrm{i}g\,\overline{\phi}_J\, a_{JI\mu} - \overline{b}_{J\mu}\, M_{IJ}.
\end{aligned}$$

In matrix notation, the gauge-invariant Lagrangian (42) thus writes

$$\begin{aligned}
\mathcal{L}_{3,\mathrm{KG}} \;&= \left(\tfrac{\partial \overline{\phi}}{\partial x_\alpha} + \mathrm{i}g\,\overline{\phi}\,\boldsymbol{a}^\alpha - \overline{\boldsymbol{b}}^{\,\alpha} M^T\right)\left(\tfrac{\partial \phi}{\partial x^\alpha} - \mathrm{i}g\,\boldsymbol{a}_\alpha \phi - M\boldsymbol{b}_\alpha\right) \\
&\quad - \mathrm{Tr}\left(\tfrac{1}{4}\boldsymbol{f}^{\alpha\beta}\boldsymbol{f}_{\alpha\beta}\right) - \tfrac{1}{2}\overline{\boldsymbol{q}}^{\,\alpha\beta}\boldsymbol{q}_{\alpha\beta}
\end{aligned}$$

with

$$\begin{aligned}
\boldsymbol{f}_{\mu\nu} &= \tfrac{\partial \boldsymbol{a}_\nu}{\partial x^\mu} - \tfrac{\partial \boldsymbol{a}_\mu}{\partial x^\nu} + \mathrm{i}g\,(\boldsymbol{a}_\nu\,\boldsymbol{a}_\mu - \boldsymbol{a}_\mu\,\boldsymbol{a}_\nu) \\
M\boldsymbol{q}_{\mu\nu} &= M\left(\tfrac{\partial \boldsymbol{b}_\nu}{\partial x^\mu} - \tfrac{\partial \boldsymbol{b}_\mu}{\partial x^\nu}\right) + \mathrm{i}g\left(\boldsymbol{a}_\nu M\boldsymbol{b}_\mu - \boldsymbol{a}_\mu M\boldsymbol{b}_\nu + \boldsymbol{f}_{\mu\nu}\phi\right) \\
\overline{\boldsymbol{q}}_{\mu\nu} M^T &= \left(\tfrac{\partial \overline{\boldsymbol{b}}_\nu}{\partial x^\mu} - \tfrac{\partial \overline{\boldsymbol{b}}_\mu}{\partial x^\nu}\right) M^T - \mathrm{i}g\left(\overline{\boldsymbol{b}}_\mu M^T \boldsymbol{a}_\nu - \overline{\boldsymbol{b}}_\nu M^T \boldsymbol{a}_\mu + \overline{\phi}\,\boldsymbol{f}_{\mu\nu}\right).
\end{aligned}$$

The terms in parentheses in the first line of $\mathcal{L}_{3,\mathrm{KG}}$ can be regarded as the "minimum coupling rule" for the actual system. Under the inhomogeneous transformation prescription of the base fields from Eqs. (12) and the transformation rules of the gauge fields from Eqs. (20), the Lagrangian $\mathcal{L}_{3,\mathrm{KG}}$ is form-invariant. Moreover, the Lagrangian contains a term that is proportional to the square of the 4-vector gauge fields $\boldsymbol{b}_J$

$$\overline{\boldsymbol{b}}^{\,\alpha} M^T M\, \boldsymbol{b}_\alpha,$$

which represents a Proca mass term for an N-tuple of possibly charged bosons. Setting up the Euler-Lagrange equation for the gauge fields $\boldsymbol{b}_\mu$, we get

$$\frac{\partial \boldsymbol{q}^{\mu\alpha}}{\partial x^\alpha} - \mathrm{i}gM^T\boldsymbol{a}_\alpha\big(M^T\big)^{-1}\boldsymbol{q}^{\mu\alpha} + M^T\left(\frac{\partial \phi}{\partial x_\mu} - \mathrm{i}g\,\boldsymbol{a}^\mu\phi\right) - M^T M\,\boldsymbol{b}^\mu = 0.$$

We observe that this equation describes an N-tuple *massive* bosonic fields $b_{J\mu}$, in conjunction with their interactions with the *massless* gauge fields $a_{IJ\mu}$ and the base fields, ϕ_I.

Expanding the last term of the Lagrangian (42), we can separate this Lagrangian into a renormalizable $\mathcal{L}^{\mathrm{r}}_{3,\mathrm{KG}}$ part

$$\begin{aligned}
\mathcal{L}^{\mathrm{r}}_{3,\mathrm{KG}} &= \overline{\pi}_I^{\,\alpha}\, \pi_{I\alpha} - \tfrac{1}{4} f_{JK}^{\alpha\beta}\, f_{KJ\alpha\beta} - \tfrac{1}{2}\overline{h}_J^{\,\alpha\beta} h_{J\alpha\beta} \\
h_{J\mu\nu} &= \tfrac{\partial b_{J\nu}}{\partial x^\mu} - \tfrac{\partial b_{J\mu}}{\partial x^\nu} + \mathrm{i}g\,\tilde{M}_{JI}\,(a_{IK\nu}\, M_{KL}\, b_{L\mu} - a_{IK\mu} M_{KL}\, b_{L\nu}) \\
\overline{h}_{J\mu\nu} &= \tfrac{\partial \overline{b}_{J\nu}}{\partial x^\mu} - \tfrac{\partial \overline{b}_{J\mu}}{\partial x^\nu} - \mathrm{i}g\left(\overline{b}_{L\mu}\, M_{KL}\, a_{KI\nu} - \overline{b}_{L\nu}\, M_{KL}\, a_{KI\mu}\right)\tilde{M}_{JI},
\end{aligned}$$

and into a non-renormalizable $\mathcal{L}^{\mathrm{nr}}_{3,\mathrm{KG}}$ part

$$\begin{aligned}
&\mathcal{L}^{\mathrm{nr}}_{3,\mathrm{KG}} = \\
&\tfrac{1}{2}\mathrm{i}g\left[\left(\tfrac{\partial \overline{b}_{J\beta}}{\partial x^\alpha} - \tfrac{\partial \overline{b}_{J\alpha}}{\partial x^\beta}\right)\tilde{M}_{JI} f^{\alpha\beta}_{IK}\phi_K - \overline{\phi}_K f^{\alpha\beta}_{KI}\tilde{M}_{JI}\left(\tfrac{\partial b_{J\beta}}{\partial x^\alpha} - \tfrac{\partial b_{J\alpha}}{\partial x^\beta}\right)\right] \\
&\quad + \tfrac{1}{2}\left(\tfrac{g}{\det M}\right)^2\Big[\left(\overline{b}_{L\alpha}\, M_{KL}\, a_{KI\beta} - \overline{b}_{L\beta}\, M_{KL}\, a_{KI\alpha}\right) f^{\alpha\beta}_{IJ}\phi_J \\
&\quad + \overline{\phi}_K f^{\alpha\beta}_{KI}\left(a_{IL\beta} M_{LJ}\, b_{J\alpha} - a_{IL\alpha}\, M_{LJ}\, b_{J\beta}\right) + \overline{\phi}_K f^{\alpha\beta}_{KI} f_{IJ\alpha\beta}\phi_J\Big].
\end{aligned}$$

The first line vanishes if we restrict ourselves to *real* fields. $\mathcal{L}^{\mathrm{nr}}_{3,\mathrm{KG}}$ vanishes completely if $g = 0$, hence if all couplings to the massless gauge fields $\boldsymbol{a}_{IK}$ are skipped. This corresponds to a pure shifting transformation that is generated by Eq. (13) with $U = \mathbb{1}$.

For the case $N = 1$, hence for a single base field ϕ, the following twofold amended Klein-Gordon Lagrangian $\mathcal{L}_{3,\mathrm{KG}}$

$$\begin{aligned}
\mathcal{L}_{3,\mathrm{KG}} = &\left(\tfrac{\partial \overline{\phi}}{\partial x_\alpha} + \mathrm{i}g\,\overline{\phi}\, a^\alpha - m\,\overline{b}^{\,\alpha}\right)\left(\tfrac{\partial \phi}{\partial x^\alpha} - \mathrm{i}g\, a_\alpha\phi - m\, b_\alpha\right) \\
&-\tfrac{1}{4} f^{\alpha\beta}\, f_{\alpha\beta} - \tfrac{1}{2}\overline{q}^{\,\alpha\beta} q_{\alpha\beta}
\end{aligned}$$

is form-invariant under the combined local gauge transformation

$$\begin{aligned}
\phi \mapsto \Phi \quad &= \phi\, e^{\mathrm{i}\Lambda} + \varphi, \qquad a_\mu \mapsto A_\mu = a_\mu + \tfrac{1}{g}\tfrac{\partial \Lambda}{\partial x^\mu} \\
b_\mu \mapsto B_\mu \quad &= b_\mu\, e^{\mathrm{i}\Lambda} - \tfrac{\mathrm{i}g}{m}\left(a_\mu + \tfrac{1}{g}\tfrac{\partial \Lambda}{\partial x^\mu}\right)\varphi + \tfrac{1}{m}\tfrac{\partial \varphi}{\partial x^\mu}.
\end{aligned}$$

The field tensors then simplify to

$$\begin{aligned}
f_{\mu\nu} \quad &= \tfrac{\partial a_\nu}{\partial x^\mu} - \tfrac{\partial a_\mu}{\partial x^\nu} \\
q_{\mu\nu} \quad &= \tfrac{\partial b_\nu}{\partial x^\mu} - \tfrac{\partial b_\mu}{\partial x^\nu} + \mathrm{i}g\,(a_\nu\, b_\mu - a_\mu\, b_\nu) + \tfrac{\mathrm{i}g}{m}\left(\tfrac{\partial a_\nu}{\partial x^\mu} - \tfrac{\partial a_\mu}{\partial x^\nu}\right)\phi \\
\overline{q}_{\mu\nu} \quad &= \tfrac{\partial \overline{b}_\nu}{\partial x^\mu} - \tfrac{\partial \overline{b}_\mu}{\partial x^\nu} - \mathrm{i}g\left(\overline{b}_\mu\, a_\nu - \overline{b}_\nu\, a_\mu\right) - \tfrac{\mathrm{i}g}{m}\overline{\phi}\left(\tfrac{\partial a_\nu}{\partial x^\mu} - \tfrac{\partial a_\mu}{\partial x^\nu}\right).
\end{aligned}$$

With $m^2\,\overline{b}^{\,\alpha} b_\alpha$, this locally gauge-invariant Lagrangian contains a mass term for the complex bosonic 4-vector gauge field b_μ. The subsequent equation for the *massive* gauge field b_μ is thus

$$\frac{\partial q^{\mu\alpha}}{\partial x^\alpha} - \mathrm{i}g\, a_\alpha q^{\mu\alpha} + m\left(\frac{\partial \phi}{\partial x_\mu} - \mathrm{i}g\, a^\mu\phi\right) - m^2 b^\mu = 0.$$

From the transformation rule for the fields, the rule for the momenta $Q_{\mu\nu}$ follows as

$$Q_{\mu\nu} = q_{\mu\nu}\, e^{\mathrm{i}\Lambda(x)}.$$

It is then easy to verify that the field equation is indeed form-invariant under the above combined local transformation of the fields ϕ, a_μ, b_μ.

The Lagrangian $\mathcal{L}_{3,\mathrm{KG}}$ can again be split into a renormalizable part $\mathcal{L}^{\mathrm{r}}_{3,\mathrm{KG}}$

$$\begin{aligned}
\mathcal{L}^{\mathrm{r}}_{3,\mathrm{KG}} \quad = &\left(\tfrac{\partial \overline{\phi}}{\partial x_\alpha} + \mathrm{i}g\,\overline{\phi}\, a^\alpha - m\,\overline{b}^{\,\alpha}\right)\left(\tfrac{\partial \phi}{\partial x^\alpha} - \mathrm{i}g\, a_\alpha\phi - m\, b_\alpha\right) \\
&-\tfrac{1}{4} f^{\alpha\beta}\, f_{\alpha\beta} - \tfrac{1}{2}\overline{h}^{\,\alpha\beta} h_{\alpha\beta}
\end{aligned}$$

$$\begin{aligned}
f_{\mu\nu} \quad &= \tfrac{\partial a_\nu}{\partial x^\mu} - \tfrac{\partial a_\mu}{\partial x^\nu} \\
h_{\mu\nu} \quad &= \tfrac{\partial b_\nu}{\partial x^\mu} - \tfrac{\partial b_\mu}{\partial x^\nu} + \mathrm{i}g\,(a_\nu\, b_\mu - a_\mu\, b_\nu) \\
\overline{h}_{\mu\nu} \quad &= \tfrac{\partial \overline{b}_\nu}{\partial x^\mu} - \tfrac{\partial \overline{b}_\mu}{\partial x^\nu} - \mathrm{i}g\left(\overline{b}_\mu\, a_\nu - \overline{b}_\nu\, a_\mu\right)
\end{aligned}$$

and a non-renormalizable part $\mathcal{L}^{\mathrm{nr}}_{3,\mathrm{KG}}$,

$$\mathcal{L}^{\mathrm{nr}}_{3,\mathrm{KG}} = \frac{\mathrm{i}g}{m}\left(\overline{h}_{\,\alpha\beta}\phi - \overline{\phi}\, h_{\alpha\beta} - \frac{\mathrm{i}g}{m}\overline{\phi}\phi\, f_{\alpha\beta}\right) f^{\alpha\beta}.$$

Conclusions

With the present paper, we have worked out a complete non-Abelian theory of *inhomogeneous* local gauge transformations. The theory was worked out as a canonical transformation in the realm of covariant Hamiltonian field theory. A particularly useful device was the definition of a gauge field *matrix* $\boldsymbol{a}_{IJ}$, with each matrix element representing a 4-vector gauge field. This way, the mutual interactions of base fields ϕ_I and both sets of gauge fields, $\boldsymbol{a}_{IJ}$ and $\boldsymbol{b}_J$, attain a straightforward algebraic representation as ordinary matrix products.

Not a single assumption or postulate needed to be incorporated in the course of the derivation. Moreover, no premise with respect to a particular "potential energy" function was required nor any draft on a "symmetry breaking" mechanism. The only restriction needed to render the theory consistent was to require the *mass matrix* to be *orthogonal*.

Requiring a theory to be form-invariant under the SU(N) gauge group generally enforces all gauge fields to be *massless*. Yet, we are free to define *other* local gauge groups, under which we require the theory to be form-invariant. Defining a local *shifting* transformations of the base fields means to submit the given system to the action of *fluctuating background fields*. A local gauge invariance of the system's Hamiltonian then actually requires the existence of *massive gauge fields*. Specifically, the formalism enforces to introduce both a set of massless gauge fields and a set of massive gauge fields.

The various mutual interactions of base and gauge fields that are described by the corresponding gauge-invariant Lagrangian $\mathcal{L}_3$ give rise to a variety of processes that can be used to test whether this beautiful formalism is actually reflected by nature.

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An approach of Spin Hall Effect of light

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Abstract: *It has been recently found that the equations of motion of several semiclassical systems must take into account anomalous velocity term arising from Berry phase contributions. Those terms are for instance responsible for the spin Hall effect in semiconductors or the gravitational birefringence of photons propagating in a static gravitational field. Intensive ongoing research on this subject seems to indicate that a broad class of quantum systems might have their dynamics affected by Berry phase terms. In this article we discuss some aspects of this phenomenon with a focus on Dirac particles and photons.*

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Introduction

It has been recently found that light exhibits a spin Hall effect similar to the the spin Hall effect occurring for charge carriers in solid-state systems. This shows the universality of the comportment of particles of different nature. In a medium with position dpendent refractive index, the center of a beam of light undergoes a spin-dependent displacement perpendicular both to the initial propagation direction and to the change in the propagation direction. Two different spin (helicity) components acqu opposite displacements. Therefore, a linearly polarized light will slightly splits into two beams, each containing different helicity states. Many authors have shown the importance of the Berry's phase for the understanding of this phenomenon. Indeed, Berry's phase provides a geometrical frame that describes the light polarization variations along the propagation trajectory and the additional term which appears in the equations of motion leading to the spin Hall effect of light. We can connecte this effect with the back reaction of spin (rapid variable) on the trajectory (slow variable) [1]- [18].

In this proceeding we want to introduce a pedestrian approach of this effect coming from Feynmann brackets which tell us how to play with the different symmetries laws and with the purpose of introducing the Berry phase in the formalism of the Noncommutative Quantum Mechanics (NCQM). The paper is organized as follows. We begin by briefly introducing the formalism of Feynman's brackets, then from the approach of the NCQM we show how to retrieve the spin Hall effect of spinning particules. As an application we consider the photon propagating in a gravitationnal field.

Feynman brackets

Feynman's proof of Maxwell equation

A remarkable way to solve Maxwell's equations is exposed in an old unpublished work of Feynman, reported in an elegant paper by Dyson [19] published in 1990. Initial Feynman's motivation was to develop a quantization procedure without resort to a Lagrangian or a Hamiltonian. For this goal let consider a non relativistic particle of mass m submitted to an external force: $m\frac{d\dot{x}^i}{dt} = F^i(\mathbf{x}, \dot{\mathbf{x}}, t)$ and the fibber tangent space with a symplectic structure defined by the " Feynman brackets " $\left[x^i, x^j\right] = 0$ and $\left[x^i, \dot{x}^j\right] = \delta^{ij}$.

From only the assumptions of Leibnitz law and Jacobi identity Feynman, in 1948, deduced the following relations

$$\left[\dot{x}^i, \dot{x}^j\right] = F^{ij}(\mathbf{x},t) = \varepsilon^{ijk}B_k(\mathbf{x},t); F^i(\mathbf{x},\dot{\mathbf{x}},t)$$
$$= E^i(\mathbf{x},t) + \varepsilon^{ijk}v_j B_k(\mathbf{x},t); div\overrightarrow{B} = 0 \text{ and } \overrightarrow{rot}\overrightarrow{E} = -\frac{\overrightarrow{\partial B}}{\partial t} \tag{1}$$

So(3) symmetry

In order to study the symmetry breaking of the sO(3) algebra we use the usual angular momentum $L^i = m\varepsilon^i{}_{jk}x^j\dot{x}^k$ which is a constant of motion in absence of gauge field. In fact, no electromagnetic field implies $F^{ij}(\mathbf{x}) = \varepsilon^{ijk}B_k(\mathbf{x}) = \left[\dot{x}^i, \dot{x}^j\right] = 0$, and the expression of the sO(3) Lie algebra with our brackets gives then the standard algebra with the presence of an electromagnetic field

$$\left[x^i, L^j\right] = \varepsilon^{ij}{}_k x^k; \left[\dot{x}^i, L^j\right] = \varepsilon^{ij}{}_k\dot{x}^k + \frac{q}{m}\varepsilon^j{}_{kl}x^k F^{il}(\mathbf{x}); \left[L^i, L^j\right]$$
$$= \varepsilon^{ij}{}_k L^k + q\varepsilon^i{}_{kl}\varepsilon^j{}_{ms}x^k x^m F^{ls}(\mathbf{x})..$$
(2)

In order to restore this sO(3) algebra we introduce a new angular momentum $M^i(\mathbf{X})$ which is *a priori* position and velocity dependent. We consider then the following transformation law $L^i(\mathbf{X}) \to \mathcal{L}^i(\mathbf{X}) = L^i(\mathbf{X}) + M^i(\mathbf{X})$, and we require that this new angular momentum $\mathcal{L}^i$ verifies the usual sO(3) algebra. We can easily deduce three constrains for $M^i(\mathbf{X})$: is velocity independent and obeys two following relations

$$\left[\dot{x}^i, M^j\right] = -\frac{1}{m}\frac{\partial M^j(\mathbf{x})}{\partial x_i} = -\frac{q}{m}\varepsilon^j{}_{kl}x^k F^{il}(\mathbf{x}); M^i$$
$$= \frac{1}{2}q\varepsilon_{jkl}x^i x^k F^{jl}(\mathbf{x}) = -q\left(\mathbf{x}\cdot\mathbf{B}\right)x^i. \tag{3}$$

which are compatible if the magnetic field $\mathbf{B}$ is the Dirac magnetic monopole field $\mathbf{B} = \frac{g}{4\pi}\frac{\mathbf{x}}{\|\mathbf{x}\|^3}$.

The vector $\mathbf{M}$ allowing us to restore the sO(3) symmetry is then the Poincaré momentum [20] $\mathbf{M} = -\frac{qg}{4\pi}\frac{\mathbf{x}}{\|\mathbf{x}\|}$ already found in a previous paper [21] [22]. The total angular momentum is then $\mathcal{L} = \mathbf{L} - \frac{qg}{4\pi}\frac{\mathbf{x}}{\|\mathbf{x}\|}$.

Noncommutative quantum mechanics

From Feynman to noncommutative quantum mechanics

Let now the momentum vector $\mathbf{p}$ replace the velocity vector $\dot{\mathbf{x}}$ in the Feynman formalism. Consider a quantum particle of mass m whose coordinates satisfy the deformed Heisenberg algebra

$$\left[x^i, x^j\right] = i\hbar q_\theta\theta^{ij}(\mathbf{x},\ \mathbf{p}), \quad \left[x^i, p^j\right] = i\hbar\delta^{ij}, \quad \left[p^i, p^j\right] = 0, \tag{4}$$

where θ is a field which is *a priori* position and momentum dependent and q_θ is a charge characterizing the intensity of the interaction of the particle and the θ field. The commutation of the momentum implies that there is no external magnetic field. It is well known that these commutation relations can be obtained from the deformation of the Poisson algebra of classical observable with a provided Weyl-Wigner-Moyal product [23] expanded at the first order in θ. In a previous paper [22] we generalized the quantum mechanics in noncommutative geometry by considering a quantum particle of mass m whose coordinates satisfy the deformed Heisenberg algebra $\left[x^i, x^j\right] = i\hbar\theta^{ij}(\mathbf{x},\mathbf{p})$, $[x^i, p^j] = i\hbar\delta^{ij}$, and $[p^i, p^j] = 0$. From the Jacobi identity $[p^i, \left[x^j, x^k\right]] + [x^j, [x^k, p^i]] + [x^k, [p^i, x^j]] = 0$, we deduced the important property that the θ field is only momentum dependent. Note that instead of this momentum dependent field, other authors (see [24] and references therein) studied gauge theory with a position dependent field $\theta(\mathbf{x})$ using a Konsevich [25] product which is a generalization of the Moyal product [23]. An important consequence of the noncommmutativity between the coordinates is that neither the position operator does satisfy the usual law $[x^i, L^j] = i\hbar\varepsilon^{ijk}x_k$, nor the angular momentum satisfy the standard $SO(3)$ algebra $[L^i, L^j] = i\hbar\varepsilon^{ijk}L_k$. Actually we have $[x^i, L^j] = i\hbar\varepsilon^{ijk}x_k + i\hbar\varepsilon^j{}_{kl}p^l\theta^{ik}(\mathbf{p})$,

and $[L^i, L^j] = i\hbar\varepsilon^{ij}{}_k L^k + i\hbar\varepsilon^i{}_{kl}\varepsilon^j{}_{mn} p^l p^n \theta^{km}(\mathbf{p})$. To remedy this absence of generators of rotations in the noncommutative geometry we had to introduce a generalized angular momentum $\mathbf{J} = \mathbf{r}\wedge\mathbf{p} + \lambda\frac{\mathbf{p}}{p}$, that satisfies the $SO(3)$ algebra. The position operator then transforms as a vector under rotations i.e., $[x^i, J^j] = i\hbar\varepsilon^{ijk}x_k$. The presence of the dual Poincare momentum $\lambda\mathbf{p}/p$ leads to a dual Dirac monopole in momentum space for the position algebra $\left[x^i, x^j\right] = -i\hbar\lambda\varepsilon^{ijk}\frac{p^k}{p^3}$. This result immediately implies that the coordinates of spinless particles are commuting. Another consequence is the quantification of the helicity $\lambda = n\hbar/2$ that arises from the restoration of the translation group of momentum that is broken by the monopole [26].

Link between Berry phase and noncommutative quantum mechanics

Previously we have postulated noncommutativity between the components of the position operator, now we will show that it can emerge naturally in some systems. To see this consider the Dirac Hamiltonian of a massive particle submitted to a potential $\hat{H} = \alpha.\mathbf{p} + \beta m + \hat{V}(\mathbf{R})$, where $\hat{V}$ is an operator that acts only on the orbital degrees of freedom. Using the Foldy-Wouthuysen unitary transformation$U(\mathbf{p}) = \frac{E_p + mc^2 + c\beta\alpha.\mathbf{p}}{\sqrt{2E_p(E_p+mc^2)}}$, with $E_p = \sqrt{p^2c^2 + m^2c^4}$, we get the following transformed Hamiltonian$U(\mathbf{p})\hat{H}U(\mathbf{p})^+ = E_p\beta + U(\mathbf{p})\hat{V}(i\hbar\partial_{\mathbf{p}})U(\mathbf{p})^+$.The kinetic energy is now diagonal whereas the potential term becomes $\hat{V}(\mathbf{D})$ with the covariant derivative defined by $\mathbf{D} = i\hbar\partial_{\mathbf{p}} + \mathbf{A}$, and with the gauge potential $\mathbf{A} = i\hbar U(\mathbf{p})\partial_{\mathbf{p}}U(\mathbf{p})^+$, which reads

$$\mathbf{A} = \frac{\hbar c\left(ic^2\mathbf{p}(\alpha.\mathbf{p})\beta + i\beta\left(E_p + mc^2\right)E_p\alpha - cE_p\ \mathbf{\Sigma}\wedge\mathbf{p}\right)}{2E_p^2\left(E_p + mc^2\right)}, \tag{5}$$

where $\mathbf{\Sigma} = 1\otimes\sigma$, is a (4×4) matrix. We consider now the adiabatic approximation which consists to neglect the interband transition. We then keep only the bloc diagonal matrix element in the gauge potential and project on the subspace of positive energy. Then we obtain the Berry connection, the 2×2 matrix $\mathbf{A}(\mathbf{p}) = i\hbar\mathcal{P}(U\partial_{\mathbf{p}}U^+)$, where $\mathcal{P}$ is a projector on the positive energy subspace. In this context the θ field we postulated in [12] appears naturally as a consequence of the adiabatic motion of a Dirac particle and corresponds to a non-Abelian gauge curvature satisfying the relation $\theta^{ij}(\mathbf{p},\sigma) = \partial_{p^i}A^j - \partial_{p^j}A^i + \left[A^i, A^j\right]$. This projection cancels the zitterbewegung which corresponds to an oscillatory movement around the mean position of the particle that mixes the positive and negative energies. In this way we obtain a non trivial gauge connection allowing us to define a new position operator $\mathbf{r}$ for this particle$\mathbf{r} = i\hbar\partial_{\mathbf{p}} + \frac{c^2\hbar(\ \mathbf{p}\wedge\sigma)}{2E_p(E_p+mc^2)}$, which is a (2×2) matrix. For a different work with operator valued position connected to the spin-degree of freedom see [27]. Zitterbewegung-free noncommutative coordinates were also introduced for massless particle with rigidity and in the context of anyons [28]. The commutation relations between the coordinates are then $\left[x^i, x^j\right] = i\hbar\theta^{ij}(\mathbf{p},\sigma) = -i\hbar^2\varepsilon_{ijk}\frac{c^4}{2E_p^3}\left(m\sigma^k + \frac{p^k(\mathbf{p}.\sigma)}{E_p+mc^2}\right)$.

To generalize the construction of the position operator for a particle with unspecified $n/2$ $(n>1)$ spin, we can use the Bargmann-Wigner equations and obtain

$$\left[x^i, x^j\right] = i\hbar\theta^{ij}(\mathbf{p},\mathbf{S}) = -i\hbar\varepsilon_{ijk}\frac{c^4}{E_p^3}\left(mS^k + \frac{p^k(\mathbf{p}.\mathbf{S})}{E_p + mc^2}\right) \tag{6}$$

For a massless particle we get the relation $\mathbf{r} = i\hbar\partial_{\mathbf{p}} + \mathbf{p}\wedge\mathbf{S}/p^2$, with the commutation relation giving rise to the monopole $\left[x^i, x^j\right] = i\hbar\theta^{ij}(\mathbf{p}) = -i\hbar\varepsilon_{ijk}\lambda\frac{p^k}{p^3}$. It is not surprising that a massless particle has a monopole Berry curvature as it is well known that the band touching point acts as a monopole in momentum space [5]. This is precisely the case for massless particles for which the positive and negative energy band are degenerate in $p=0$. The monopole appears as a limiting case of a more general non abelian Berry curvature arising from an adiabatic process of massive spinning particles.

The equation of motion of a particle in a arbitrary potential is then given by

$$\dot{\mathbf{r}} = \frac{\mathbf{p}}{E_p} - \dot{\mathbf{p}}\wedge\theta \tag{7}$$

with $\dot{\mathbf{p}} = -\nabla V(\mathbf{r})$ with $\theta^i = \varepsilon^{ijk}\theta_{jk}/2$. Term $\dot{\mathbf{p}}\wedge\theta$ is called the anomalous velocity.

Spin Hall effect of light

An exemple of topological spin transport is the ultrarelativistic limit. Experimentally a topological spin transport has been already observed in the case of the photon propagation in an inhomogeneous medium [29], where the right and left circular polarization propagate along different trajectories in a wave guide (the transvers shift is observable due to the multiple reflexions), a phenomena interpreted quantum mechanically as arising from the interaction between the orbital momentum and the spin of the photon [29]. To interpret the experiments these authors introduced a compicate phenomenological Hamiltonian. Our approach provides a new satisfactory interpretation as this effect also called optical Magnus effect is now explained in terms of the non-commutative property of the position operator that contains the spin-orbit interaction. In this sens, this effect is just the ultra-relativistic spin-Hall effect. Note that the adiabaticity criteria has been proved to be valid in [3]. To illustrate our purpose consider the simple photon Hamiltonian in the inhomogeneous medium $H = pc/n(r)$. The equations of motion $\dot{x} = \frac{1}{i\hbar}[x, H]$ and $\dot{p} = \frac{1}{i\hbar}[p, H]$ in the semi-classical approximation leads to following relation between velocity and momentum $\frac{dx^i}{dt} = \frac{c}{n}\left(\frac{p^i}{p} + \frac{\lambda \varepsilon^{ijk} p_k}{p^2}\frac{\partial \ln n}{\partial x^j}\right)$ which containes an unusal contribution due to the Berry phase. As a consequence the velocity is no more equal to c/n. These last equations are the same as those introduced phenomenogicaly in [29], but here are deduced rigoursely from different physical consideration. Similar equations are also given in [10] where the optical Magnus effect is also interpreted in terms of a monopole Berry curvature but in the context of geometrical optic. Our theory is generalizable to the photon propagation in a non isotrop medium, a situation which is mentioned in [29] but could not be studied with their phenomenological approach.

Photon in a static gravitational field.

We now apply our general approach to the case of a photon propagating in an arbitrary static gravitational field, where $g_{0i} = 0$ for $i = 1, 2, 3$, so that $ds^2 = g_{00}(dx^0)^2 - g_{ij}dx^i dx^j = 0$. As explained in [31] the photon description is obtained by considering first a Dirac massless particle (massless neutrino) and then by replacing the Pauli matrices σ by the spin-1 matrices $\mathbf{S}$. Therefore we start with the Dirac Hamiltonian in static gravitational field which can be written $\hat{H} = \sqrt{g_{00}}\alpha.\tilde{\mathbf{P}} + \frac{\hbar}{4}\varepsilon_{\varrho\beta\gamma}\Gamma_0^{\varrho\beta}\sigma^\gamma + i\frac{\hbar}{4}\Gamma_0^{0\beta}\alpha_\beta$ with $\tilde{\mathbf{P}}$ given by $\tilde{P}_\alpha = h^i_\alpha(\mathbf{R})(P_i + \frac{\hbar}{4}\varepsilon_{\varrho\beta\gamma}\Gamma_i^{\varrho\beta}\sigma^\gamma)$ with h^i_α the static orthonormal dreibein ($\alpha = 1, 2, 3$), $\Gamma_i^{\alpha\beta}$ the spin connection components and $\varepsilon_{\alpha\beta\gamma}\sigma^\gamma = \frac{i}{8}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)$. The coordinate operator is again given by $\mathbf{R} = i\hbar\partial_{\mathbf{p}}$. Note that here we consider the general case where an arbitrary static torsion of space is allowed. It is known [32] that for a static gravitational field (which is the case considered here), the Hamiltonian $\hat{H}$ is Hermitian. We now want to diagonalize $\hat{H}$ through a unitary transformation $U(\tilde{\mathbf{P}})$. Because the components of $\tilde{\mathbf{P}}$ depend both on operators $\mathbf{P}$ and $\mathbf{R}$ the diagonalization at order $\hbar$ is performed by adapting the method detailed above to block-diagonal Hamiltonians. To do so, we first write $\hat{H}$ in a symmetrical way in $\mathbf{P}$ and $\mathbf{R}$ at first order in $\hbar$. This is easily achieved using the Hermiticity of the Hamiltonian which yields $\hat{H} = \frac{1}{2}\left(\sqrt{g_{00}}\alpha.\tilde{\mathbf{P}} + \tilde{\mathbf{P}}^+.\alpha\sqrt{g_{00}}\right) + \frac{\hbar}{4}\varepsilon_{\varrho\beta\gamma}\Gamma_0^{\varrho\beta}\sigma^\gamma$. Finally we arrive at the following expression for the diagonal positive (we have projected on the positive energy subspace) energy representation $\tilde{\varepsilon}$: $\tilde{\varepsilon} = \varepsilon + \frac{\lambda}{4}\frac{\mathbf{p}.\mathbf{\Gamma}_0}{p} + \frac{\hbar\mathbf{B}.\sigma}{2\varepsilon} - \frac{(\mathbf{A}_R\times\mathbf{p}).\mathbf{B}}{\varepsilon^{(r)}}$. where we have introduced a field $B_\gamma = -\frac{1}{2}P_\delta T^{\alpha\beta\delta}\varepsilon_{\alpha\beta\gamma}$, with $T^{\alpha\beta\delta} = h^\delta_k\left(h^{l\alpha}\partial_l h^{k\beta} - h^{l\beta}\partial_l h^{k\alpha}\right) + h^{l\alpha}\Gamma_l^{\beta\delta} - h^{l\beta}\Gamma_l^{\alpha\delta}$ the usual torsion for a static metric (where only space indices in the summations give non zero contributions). We have also defined the last equation $\varepsilon = c\sqrt{\left(p_i + \frac{\lambda}{4}\frac{\Gamma_i(\mathbf{r}).\mathbf{p}}{p}\right)g^{ij}g_{00}\left(p_j + \frac{\lambda}{4}\frac{\Gamma_j(\mathbf{r}).\mathbf{p}}{p}\right)}$, with the γ-th component of the vector $\mathbf{\Gamma}_i$ as $\Gamma_{i,\gamma} = \varepsilon_{\varrho\beta\gamma}\Gamma_i^{\varrho\beta}(\mathbf{r})$ and the helicity $\lambda = \frac{\hbar\mathbf{p}.\sigma}{p}$. Note that the dynamical operators are no $\mathbf{r} = \mathbf{R} + \hbar c^2\frac{\mathbf{P}\times\mathbf{\Sigma}}{2\varepsilon^2}$ and $\mathbf{p} = \mathbf{P} - \hbar c^2(\frac{\mathbf{P}\times\mathbf{\Sigma}}{2\varepsilon^2})\nabla_{\mathbf{R}}\tilde{\mathbf{P}}$. Interestingly, this semi-classical Hamiltonian presents formally the same form as the one of a Dirac particle in a true external magnetic field ([14], [33]). The term $\mathbf{B}.\sigma$ is responsible for the Stern-Gerlach effect, and the operator $\mathbf{L} = (\mathbf{A}_R\times\mathbf{p})$ is the intrinsic angular momentum of semiclassical particles. The same contribution appears also in the context of the semiclassical behavior of Bloch electrons (spinless) in an external magnetic field ([35], [36]) where it corresponds to a magnetization term. Because of this analogy and since $T^{\alpha\beta\delta}$ is directly related to the torsion of space through $T^{\alpha\beta\delta} = h^\delta_k h^{i\alpha}h^{j\beta}T^k_{ij}$ we call $\mathbf{B}$ a magnetotorsion field.

However, this form for the energy presents the default to involve the spin rather than the helicity. Actually one can use the property $\lambda\mathbf{p}/2p = \hbar\sigma/2 - (\mathbf{A}_R\times\mathbf{p})$ to rewrite the energy as $\widetilde{\varepsilon} \simeq \varepsilon + \frac{\lambda}{4}\frac{\mathbf{p}.\mathbf{\Gamma}_0}{p} + \frac{\lambda g_{00}}{2\varepsilon}\frac{\mathbf{B}.\mathbf{p}}{p}$.This semi-classical Hamiltonian contains, in addition to the energy term ε, new contributions due to the Berry connections. Indeed, we can see also that the helicity couples to the gravitational field through the magnetotorsion field $\mathbf{B}$

which is non-zero for a space with torsion. As a consequence, a hypothetical torsion of space may be revealed through the presence of this coupling. Note that, in agrement with [37], this Hamiltonian does not contain the spin-gravity coupling term $\mathbf{\Sigma}.\nabla g_{00}$ predicted in [38].

We can also deduce the new (non-canonical) commutations rules $\left[r^i, r^j\right] = i\hbar\Theta^{ij}_{rr}$; $\left[p^i, p^j\right] = i\hbar\Theta^{ij}_{pp}$; $\left[p^i, r^j\right] = -i\hbar g^{ij} + i\hbar\Theta^{ij}_{pr}$, where $\Theta^{ij}_{\zeta\eta} = \partial_{\zeta^i} A_{\eta^j} - \partial_{\eta^i} A_{\zeta^j} + [A_{\zeta^i}, A_{\eta^j}]$ where ζ, η.mean either r or p. An explicit computation shows that at leading order

$$\Theta^{ij}_{rr} = -\hbar c^4 \frac{(\mathbf{\Sigma}.\mathbf{p})\, p_\gamma}{2\varepsilon^4} \varepsilon^{\alpha\beta\gamma} h^i_\alpha h^j_\beta; \Theta^{ij}_{pp} = -\hbar c^4 \frac{(\mathbf{\Sigma}.\mathbf{p})\, p_\gamma}{2\varepsilon^4} \nabla_{r_i} p_\alpha \nabla_{r_j} p_\beta \varepsilon^{\alpha\beta\gamma}; \quad (8)$$

$$\Theta^{ij}_{pr} = \hbar c^4 \frac{(\mathbf{\Sigma}.\mathbf{p})\, p_\gamma}{2\varepsilon^4} \nabla_{r_i} p_\alpha h^j_\beta \varepsilon^{\alpha\beta\gamma} \quad (9)$$

From the additional commutation relations between the helicity and the dynamical operators $[r_i, \lambda] = [p_i, \lambda] = 0$ we deduce the semiclassical equations of motion $\dot{\mathbf{r}} = (1 - \Theta_{pr})\nabla_{\mathbf{p}}\tilde{\varepsilon} + \dot{\mathbf{p}}\times\Theta_{rr}$ and $\dot{\mathbf{p}} = -(1 - \Theta_{pr})\nabla_{\mathbf{r}}\tilde{\varepsilon} + \dot{\mathbf{r}} \times \Theta_{pp}$.To complete the dynamical description of the photon notice that at the leading order the helicity λ is not changed by the unitary transformation which diagonalizes the Hamiltonian so that it can be written $\lambda = \hbar\mathbf{p}.\mathbf{\Sigma}/p$. After a short computation one can check that the helicity is always conserved $\frac{d}{dt}\left(\frac{\hbar\mathbf{p}.\mathbf{\Sigma}}{p}\right) = 0$ for an arbitrary static gravitational field independently of the existence of a torsion of space.

These motion equations are the new semiclassical equations of motion for a photon in a static gravitational field. They describe the ray trajectory of light in the first approximation of geometrical optics (GO). (In GO it is common to work with dimensionless momentum operator $\mathbf{p} = k_0^{-1}\mathbf{k}$ with $k_0 = \omega/c$ instead of the momentum [34]). For zero Berry curvatures we obtain the well known zero order approximation of GO and photons follow the null geodesic. The velocity equation contains the by now well known anomalous contribution $\dot{\mathbf{p}}\times\Theta^{rr}$ which is at the origin of the intrinsic spin Hall effect (or Magnus effect) of the photon in an isotropic inhomogeneous medium of refractive index $n(r)$ ([12], [34], [39], [40]). Indeed, this term causes an additional displacement of photons of distinct helicity in opposite directions orthogonally to the ray. Consequently, we predict gravitational birefringence since photons with distinct helicities follow different geodesics. In comparison to the usual velocity $\dot{\mathbf{r}} = \nabla_{\mathbf{p}}\tilde{\varepsilon} \sim c$, the anomalous velocity term $\mathbf{v}_\perp$ is obviously small, its order $v^i_\perp \sim c\lambda\nabla_{r^j} g^{ij}$ being proportional to the wave length $\widetilde{\lambda}$.

The momentum equation presents the dual expression $\dot{\mathbf{r}} \times \Theta_{pp}$ of the anomalous velocity which is a kind of Lorentz force which being of order $\hbar$ does not influence the velocity equation at order $\hbar$. Note that similar equations of motion with dual contributions $\dot{\mathbf{p}}\times\Theta_{rr}$ and $\dot{\mathbf{r}} \times \Theta_{pp}$ were predicted for the semiclassical dynamics of spinless electrons in crystals subject to small perturbations ([36], [35]).

Symmetric gravitational field

As a simple application, consider the symmetric case $g_{00}g^{ij} = \delta^{ij}F^2(\mathbf{R})$. A typical example of such a metric is the Schwarzschild space-time in isotropic coordinates. For a symmetric metric one has $\mathbf{B}.\mathbf{p} = \mathbf{\Gamma}_0 = 0$ and the semiclassical energy reduces to $\widetilde{\varepsilon} = c(pF(\mathbf{r}) + F(\mathbf{r})p)/2$ with the dynamical variables $\mathbf{r} = \mathbf{R} + \hbar\frac{\mathbf{P}\times\mathbf{\Sigma}}{P^2}$, $\mathbf{p} = \mathbf{P}$, and the following commutation relations $[r^i, r^j] = i\hbar\Theta^{ij}_{rr} = -i\hbar\lambda\varepsilon^{ijk}p_k/p^3$, $[p^i, p^j] = 0$, $[r^i, p^j] = i\hbar g^{ij}$. As a consequence, we derive the following equations of motion $\dot{\mathbf{r}} = \nabla_{\mathbf{p}}\tilde{\varepsilon} + \dot{\mathbf{p}}\times\Theta_{rr}$ and $\dot{\mathbf{p}} = -\nabla_{\mathbf{r}}\tilde{\varepsilon}$. In the symmetric case the equations of motion become simpler than in the general case, but the gravitational birefringence is still present. These equations were already postulated (but not derived) in [**?**] to explain the Magnus effect (the different deviation of light of distinct polarization in an inhomogeneous medium of refractive index $n(r)$) observed despite its smallness in inhomogeneous isotropic optical fibers [**?**] and also discussed theoretically in less general contexts and with different approaches in several other papers ([34], [39], [40]). This case fits within our formalism since a gravitational field can be seen as an isotropic medium related to the metric through the relation $g^{ij} = \delta^{ij}n^{-1}(r)$. Therefore the gravitational birefringence predicted here is simply due to the Magnus effect as a consequence of the photon spin-orbit interaction. In particular this effect does not need a coupling between the electromagnetic field and a torsion term as proposed in [41].

We now apply the equations of motion to compute the deflection of polarized light by a star's gravitational field. A polarization independent result is expected by the Einstein's theory of gravitation which does not consider the anomalous velocity. With the Schwarzschild metric one has $F(R) = 1 - \frac{2GM}{R}$ [38] and for the equations of motion we get $\dot{\mathbf{r}} = \frac{\mathbf{p}}{p}F + \lambda\frac{2GM}{r^3}\frac{\mathbf{r}\times\mathbf{p}}{p^2}$ and $\dot{\mathbf{p}} = -2GM\frac{\mathbf{r}}{r^3}p$.

Conclusion

Some recent applications of semiclassical methods to several branches of physics have shown the relevance of Berry phases contributions to the dynamics of a quantum system. In such a context, the coordinates and momenta algebra are no longer commutative, and the dynamical equations for these variables directly include the influence of Berry phases through the parameters of noncommutativity (Berry curvatures). The noncommutativity of the coordinate is responsible of the topological spin transport of spinning particle similarly to the spin Hall effect in spintronic physics or the optical Magnus effect in optics. We discussed the effect of the Berry phase on the propagation of light in a gravitational field. We found two new semiclassical equations motion predicting that the photon does not follow the null geodesic due to its spinning nature. The reason is an anomalous velocity, responsible for the gravitational birefringence. This last result is in agreement with the modern point of view about the spinning particles evolution. Our results are not restricted to the gravitational field but also apply to systems with anisotropic refractive indices. Currently very interesting works are carried out especially in the direction of the spin Hall effect of light [**?**] and of the study of a Dirac particle in a gravitational field [43].

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Constructive description of quantum behavior

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Abstract: *Constructive analysis of quantum concepts leads to the conclusion that quantum behavior is a natural consequence of the fundamental impossibility to trace identity of indistinguishable objects in their evolution. General mathematical arguments imply that any quantum dynamics can be reduced to a sequence of permutations. Quantum phenomena, such as interferences, arise in invariant subspaces of permutation representations of the symmetry group of a system. Observable quantities can be expressed in terms of the permutation invariants. We demonstrate that for description of quantum phenomena it is sufficient to employ the cyclotomic numbers — a constructive equivalent (dense subset) of complex numbers.*

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Introduction

Physics is an empirical science. Since the time of Newton, theories that describe empirical phenomena contain various *actual infinities*, the most typical of which is *continuum*. These mathematical constructs are *not scientific* in the sense of Karl Popper: their presence in reality can neither be verified nor falsified by any rational means. The notion of continuum lies in the base of the differential calculus. The empirical content of the concept of a derivative is the hypothesis that changes in physical data for small changes of coordinates in which the data are specified, can be approximated by linear relations. This hypothesis leads to significant simplifications. In fact, almost all contemporary physical theories involve derivatives. However, approximations of this kind may lead to substantial descriptional losses. To illustrate this, let us consider the notion of a group – the most important (because groups represent coordinate changes which are bijective mappings, i.e. data rewritings without loss of information) mathematical concept in physics. Table 1 contains lists of simple continuous and finite groups. The left part of the table has been filled by two people (W. Killing and É. Cartan) in about 5 years, whereas the

Lie groups	Finite groups
4 infinite series	**16+1+1 infinite series**
A_n, B_n, C_n, D_n.	Group of Lie type
5 exceptionals	$A_n(q), B_n(q), C_n(q), D_n(q)$, $E_6(q), E_7(q), E_8(q)$,
E_6, E_7, E_8, F_4, G_2.	$F_4(q), G_2(q), {}^2A_n\left(q^2\right), {}^2D_n\left(q^2\right), {}^2E_6\left(q^2\right), {}^3D_4\left(q^3\right)$,
	${}^2B_n\left(2^{2n+1}\right)$, ${}^2F_4\left(2^{2n+1}\right)$, ${}^2G_2\left(3^{2n+1}\right)$.
	Cyclic groups of prime order $\mathbb{Z}_p$.
	Alternating groups $\mathsf{A}_n, n \geq 5$.
	26 sporadic groups
	20 subquotients of the Monster M
	$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$, J_2, Co_1, Co_2, Co_3,
	$Fi_{22}, Fi_{23}, Fi_{24}$, HS, McL, He, Suz, HN, Th,
	B, M.
	6 pariahs
	J_1, J_3, J_4, Ru, $O'N$, Ly.

Table 1: Comparison of classifications of simple groups.

right part has been completed by about a hundred people for more than 170 years [1]. All groups from the left

have direct analogs among the groups from the right but not *vice versa*. That is, any Lie group has empirically indistinguishable counterparts (groups of Lie type) among the finite groups, but there are finite groups which can not be approximated by Lie groups. Besides the superiority of discrete mathematics over continuos in the expressive power, the finite description has obvious pragmatic benefits: absence of unnecessary entities allows us to focus on the essentials of physical problems; finiteness of description is an absolute necessity for the very possibility of computer modeling of physical systems.

Construction of number systems

Traditionally, the number systems used in applications are constructed as follows. The most fundamental primordial entities are *counters*, i.e. *natural numbers* $\mathbb{N} = \{0, 1, 2, \ldots\}$. Then *integer numbers* $\mathbb{Z}$ are constructed as equivalence classes of pairs of naturals: the equivalence $(n, m) \sim (n', m') \in \mathbb{Z}$ is defined via the relation $n + m' = n' + m$, where $n, m, n', m' \in \mathbb{N}$. This is known as the *group completion* (or the *Grothendieck group*) of the abelian monoid $(\mathbb{N}, +, 0)$. Generally, an *abelian monoid* $(A, \mathbf{op}, \mathbf{id})$ is a *set* A with a commutative *binary operation* **op** and an *identity element* **id**. The *rational numbers* $\mathbb{Q}$ are obtained as the *fraction field* of the ring $\mathbb{Z}$ in a similar way. Namely, $\mathbb{Q} \setminus \{0\}$ is the group completion of the abelian monoid $(\mathbb{Z} \setminus \{0\}, \times, 1)$.

There are *constructive* and *nonconstructive* ways to add new elements to the rational numbers. Continuous mathematics is based on nonconstructive *metric completions* of the field $\mathbb{Q}$. By the Ostrowski theorem there are only two types of such completions:

- *Real numbers* $\mathbb{R}$ are (infinite) sequences of rationals converging in the metric $|x_n - x_m|$. *Rational approximations* constitute a constructive core of the notion of real number. Rounding errors in these approximations increase in long calculations.
- *p-adic numbers* $\mathbb{Q}_p$ are sequences of rationals converging in the metric

$$|x_n - x_m|_p = \left| p^v \frac{a}{b} \right|_p \equiv p^{-v},$$

  i.e., rationals x_n and x_m are close if they have large prime power as a common factor. Here constructive core is also rational approximations, but rounding errors do not increase because the p-adic approximations are ring homomorphisms.

In the constructive paradigm, a natural way of adding new elements to the rational numbers is an *algebraic extension* $\mathbb{Q}(\alpha) = \mathbb{Q}[x]/\langle F(x) \rangle$, where α is a root of an irreducible polynomial $F(x)$. The concept of algebraic extension is quite enough for obtaining all types of number systems used in applications. For example, the complex field $\mathbb{C}$ — the main number system in quantum mechanics — is *nonconstructive completion* of some algebraic extensions, e.g., $\mathbb{Q}(i) = \mathbb{Q}[x]/\langle x^2 + 1 \rangle$.

In order to avoid as far as possible the occurrence of scientific artifacts, it is important to adhere to the Occam principle. Most economical and adequate for physics approach to introduce number systems is based on two primordial concepts

- *Natural numbers* $\mathbb{N} = \{0, 1, 2, \ldots\}$. They realize the idea of *counting*.
- *Roots of unity*, i.e., abstract solutions of the *cyclotomic equation* $\mathsf{r}^{\mathcal{C}} = 1$, where $\mathcal{C}$ is a natural number called the *conductor*. Root of unity is an algebraic incarnation of the idea of *cyclicity* (*periodicity*).

From these two concepts we obtain the following constructive derivates

- The *semiring of cyclotomic naturals* $\mathbb{N}_{\mathcal{C}}$, which is the set of all linear combinations of roots of unity with *natural* coefficients. Linear combinations of roots of unity with *integer* coefficients form the *ring of cyclotomic integers* $\mathbb{Z}_{\mathcal{C}}$. The standard properties of roots of unity imply that $\mathbb{Z}_{\mathcal{C}} = \mathbb{N}_{\mathcal{C}}$ if $\mathcal{C} \geq 2$.
- *The cyclotomic field* $\mathbb{Q}_{\mathcal{C}}$ is the fraction field of the ring $\mathbb{Z}_{\mathcal{C}}$.

A $\mathcal{C}$th root of unity r is called *primitive* if its minimal non-trivial period is equal to $\mathcal{C}$. Any root of unity can be represented as a power of a primitive root. All $\mathcal{C}$th *primitive* roots of unity are roots of the unique irreducible divisor $\Phi_{\mathcal{C}}(\mathsf{r})$ of the polynomial $\mathsf{r}^{\mathcal{C}} - 1$. $\Phi_{\mathcal{C}}(\mathsf{r})$ is called the *$\mathcal{C}$th cyclotomic polynomial*. In terms of $\Phi_{\mathcal{C}}(\mathsf{r})$ the cyclotomic field can be represented as the algebraic extension $\mathbb{Q}_{\mathcal{C}} = \mathbb{Q}[\mathsf{r}]/\langle \Phi_{\mathcal{C}}(\mathsf{r}) \rangle$.

Cyclotomic fields are only a special case of algebraic extensions. However they are very important for the constructive formulation of quantum mechanics, since all eigenvalues of linear representations of a finite group are roots of unity. This follows from the fact that any linear representation is subrepresentation of some permutation representation, and — using the decomposition of a permutation into disjoint cycles — one can easily show that the characteristic polynomial of the matrix P of a permutation of N things has the form

$$\chi_{\mathrm{P}}(\mathsf{r}) = \det(\mathrm{P} - \mathsf{r}\mathrm{I}) = (\mathsf{r}-1)^{k_1}\left(\mathsf{r}^2-1\right)^{k_2}\cdots\left(\mathsf{r}^{\mathsf{N}}-1\right)^{k_{\mathsf{N}}},$$

where k_i is the number of cycles of length i in the permutation.

The complex field $\mathbb{C}$ is *nonconstructive* derivate of natural numbers and roots of unity. It is a metric completion of *any* cyclotomic field $\mathbb{Q}_{\mathcal{C}}$ with $\mathcal{C} \geq 3$.

Constructive view of Feynman path integral

According to Feynman's approach amplitude of a quantum transition from one state to another is calculated by summing the amplitudes along all possible classical paths connecting these states. Amplitude along an individual path is evaluated as a product of the amplitudes of transitions between the nearest successive states on the path. Namely, the amplitude along a path is represented as the exponential of the action along that path

$$A_{\mathrm{U}(1)} = A_0 \exp(iS) = A_0 \exp\left(i\int_0^T L dt\right).$$

The function L depending on the *first time derivatives* of states is called the *Lagrangian*. In the discrete time the exponential of the integral turns into the product: $\exp\left(i\int L dt\right) \to \mathrm{e}^{iL_{0,1}}\ldots\mathrm{e}^{iL_{t-1,t}}\ldots\mathrm{e}^{iL_{T-1,T}}$ and expression for the amplitude takes the form

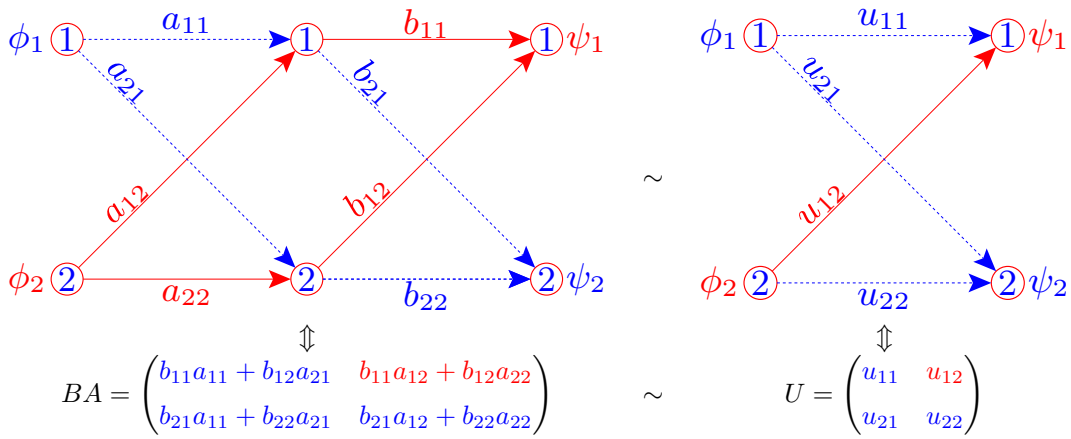
$$A_{\mathrm{U}(1)} = A_0 \mathrm{e}^{iL_{0,1}}\ldots\mathrm{e}^{iL_{t-1,t}}\ldots\mathrm{e}^{iL_{T-1,T}}.$$

We observe that factors $\mu_{t-1,t} = \mathrm{e}^{iL_{t-1,t}}$ of this product are elements of the *connection* with values in *one-dimensional* unitary representation $\mathsf{U}(1)$ of a circle, i.e. the commutative Lie group $\Gamma = S^1 \equiv \mathbb{R}/\mathbb{Z}$.

A natural generalization of this observation is to assume that the group Γ is not necessarily a circle and that its unitary representation $\rho(\Gamma)$ is not necessarily one-dimensional. In this case, the amplitude is a multi-component vector. The value of such a multicomponent amplitude on a path takes the form

$$A_{\rho(\Gamma)} = \rho(\alpha_{T,T-1})\ldots\rho(\alpha_{t,t-1})\ldots\rho(\alpha_{1,0})\,A_0, \quad \alpha_{t,t-1} \in \Gamma. \tag{1}$$

The Feynman rules, formulated in the abstract form, coincide in fact with the rules of matrix multiplication. It is clear from the illustration on which the two-step evolution of a quantum system with two states is presented in parallel in the Feynman and matrix forms:



Following the Feynman rules, the transition from, say, the state ϕ_2 to the state ψ_1 is given by the sum along two paths: $b_{11}a_{12}+b_{12}a_{22}$. But just the same expression is the element u_{12} of the product of matrices $U=BA$. The general case of arbitrary number of states and arbitrary number of time steps is easily derived from this elementary example by mathematical induction on these numbers.

In the case of non-commutative connection, formula (1) corresponds to a non-Abelian gauge theory. The above argument about the correspondence between the Feynman quantization and matrix multiplication is applicable here as well. We only need to treat the evolution matrices A, B and U as block matrices with noncommuting entries, which are matrices from the representation $\rho(\Gamma)$.

Paths in the Feynman approach can run over abstract states. Most commonly it is assumed that the paths run over space points. In what follows we will consider the problem of quantum behavior ignoring possible presence of a space in the structure of a complete set of states. We will assume that quantum evolution is determined by unitary matrices in a Hilbert space of the complete system. The possible space structure can be described in terms of tensor decompositions of the "large" Hilbert space.

Constructive core of quantum mechanics

In traditional matrix formulation quantum evolutions are described by unitary operators in a Hilbert space $\mathcal{H}$. Evolution operators U belong to a *unitary representation* of the *continuous* group $\mathrm{Aut}(\mathcal{H})$ of automorphisms of $\mathcal{H}$. To make the problem constructive we should replace the group $\mathrm{Aut}(\mathcal{H})$ by some *finite* group G which would be empirically equivalent to $\mathrm{Aut}(\mathcal{H})$.

The theory of quantum computing [2] proves the existence of *finite* sets of universal *quantum gates* that can be combined into unitary matrices which approximate to arbitrary precision any unitary operator. In other words, there exists a *finitely generated* (*countable*) group G_{FG} which is a dense subgroup of the continuous group $\mathrm{Aut}(\mathcal{H})$.

A group G is called *residually finite* [3], if for every $g \in G$, $g \neq \mathbf{1}$, there exists a homomorphism ϕ from G onto a finite group H, such that $\phi(g) \neq \mathbf{1}$. This means that any relation between the elements of G can be modeled by a relation between the elements of a finite group. Here we have an analogy with the widely used in physics trick, when the infinite space is replaced by, for example, a torus whose size is sufficient to hold the data related to a particular problem. In fact, all constructive infinite groups used in physics — a typical example is the famous braid group — are residually finite.

According to the theorem of A.I. Mal'cev [4], every finitely generated group of matrices over any field is residually finite. Thus we have the following sequence of transitions from the continuous to a finite group: $\mathrm{Aut}(\mathcal{H}) \xrightarrow{\text{approximation}} \mathsf{G}_{\mathrm{FG}} \xrightarrow{homomorphism} \mathsf{G}$.

As is well known, any linear representation of a finite group is unitary. Any representation of a finite group is a subrepresentation of some permutation representation. Let U be a representation of G in a K-dimensional Hilbert space $\mathcal{H}_{\mathsf{K}}$. Then U can be embedded into a permutation representation P of G in an N-dimensional Hilbert space $\mathcal{H}_{\mathsf{N}}$, where $\mathsf{N} \geq \mathsf{K}$. The representation P is equivalent to an action of G on a set of things $\Omega = \{\omega_1, \ldots, \omega_{\mathsf{N}}\}$ by permutations. In the proper case $\mathsf{N} > \mathsf{K}$, the embedding has the structure

$$\mathrm{T}^{-1}\mathrm{PT} = \begin{pmatrix} \mathbf{1} & & \\ & \mathrm{V} \Big\} \mathcal{H}_{\mathsf{N}-\mathsf{K}} & \\ & & \mathrm{U} \} \mathcal{H}_{\mathsf{K}} \end{pmatrix}, \quad \mathcal{H}_{\mathsf{N}} = \mathcal{H}_{\mathsf{N}-\mathsf{K}} \oplus \mathcal{H}_{\mathsf{K}},$$

where $\mathbf{1}$ is the trivial one-dimensional representation, mandatory for any permutation representation; V is a subrepresentation, which may be missing. T is a matrix of transition from the basis of the representation P to the basis in which the permutation space $\mathcal{H}_{\mathsf{N}}$ is split into the invariant subspaces $\mathcal{H}_{\mathsf{N}-\mathsf{K}}$ and $\mathcal{H}_{\mathsf{K}}$. For brevity, we will refer to this basis as "quantum basis". The data in the spaces $\mathcal{H}_{\mathsf{K}}$ and $\mathcal{H}_{\mathsf{N}-\mathsf{K}}$ are *independent* since both spaces are invariant subspaces of $\mathcal{H}_{\mathsf{N}}$. So we can consider the data in $\mathcal{H}_{\mathsf{N}-\mathsf{K}}$ as "hidden parameters" with respect to the data in $\mathcal{H}_{\mathsf{K}}$.

A trivial approach would be to set arbitrary (e.g., zero) data in the complementary subspace $\mathcal{H}_{\mathsf{N}-\mathsf{K}}$. This approach is not interesting since it is not falsifiable by means of standard quantum mechanics. In fact, it leads to standard quantum mechanics *modulo* the empirically unobservable distinction between the "finite" and the "infinite". The only difference is technical: we can replace the linear algebra in the K-dimensional space $\mathcal{H}_{\mathsf{K}}$ by permutations of N things.

A more promising approach requires some changes in the concept of quantum amplitudes. We assume [5–7] that quantum amplitudes are projections onto invariant subspaces of vectors of multiplicities ("occupation numbers") of elements of the set Ω on which the group G acts by permutations. The vectors of multiplicities $|n\rangle = \begin{pmatrix} n_1 \\ \vdots \\ n_{\mathsf{N}} \end{pmatrix}$ are elements of the *module* $\mathsf{H}_{\mathsf{N}} = \mathbb{N}^{\mathsf{N}}$, where $\mathbb{N}$ is the semiring of natural numbers. Initially we deal with the natural permutation representation of G in the module H_{N}. Using the fact that any element of a finite group has finite order we can turn the module H_{N} into the Hilbert space $\mathcal{H}_{\mathsf{N}}$. It is sufficient just to add $\mathcal{C}$th roots of unity to the natural numbers to form the *semiring* $\mathbb{N}_{\mathcal{C}}$. The conductor $\mathcal{C}$ depends on the structure of the group G. As the conductor we can always take the *exponent* of G, which is defined as the least common multiple of the orders of elements of G. However, in many cases it is sufficient to take some nontrivial divisor of the exponent. We will always assume that $\mathcal{C} \geq 2$. In this case the semiring $\mathbb{N}_{\mathcal{C}}$ becames a *ring of cyclotomic integers*. To complete the conversion of the module H_{N} into the Hilbert space $\mathcal{H}_{\mathsf{N}}$, we introduce the *cyclotomic field* $\mathbb{Q}_{\mathcal{C}}$ as a field of fractions of the ring $\mathbb{N}_{\mathcal{C}}$. If $\mathcal{C} \geq 3$, then $\mathbb{Q}_{\mathcal{C}}$ is a dense subfield of the field of complex numbers $\mathbb{C}$. In fact, algebraic properties of elements of $\mathbb{Q}_{\mathcal{C}}$ are quite sufficient for all our purposes — for example, complex conjugation corresponds to the transformation $\mathsf{r}^k \to \mathsf{r}^{\mathcal{C}-k}$ for roots of unity — so we can forget the possibility to embed $\mathbb{Q}_{\mathcal{C}}$ into $\mathbb{C}$ (as well as the very existence of the field $\mathbb{C}$).

The connection between mathematical description and observation is provided by the Born rule: the probability to register a particle described by the amplitude $|\psi\rangle$ by an apparatus configured for the amplitude $|\phi\rangle$ is

$$\mathbf{P}(\phi,\psi) = \frac{|\langle\phi \mid \psi\rangle|^2}{\langle\phi \mid \phi\rangle \langle\psi \mid \psi\rangle} \ .$$

In the "finite" background the only reasonable interpretation of probability is the *frequency interpretation*: the probability is the ratio of the number of "favorable" combinations to the total number of combinations. So we expect that $\mathbf{P}(\phi,\psi)$ must be a *rational number* if everything is arranged correctly.

Thus, in our approach the usual *non-constructive* contraposition — *complex numbers* as intermediate values against *real numbers* as observables — is replaced by the *constructive* one — *irrationalities* against *rationals*. From the constructive point of view, there is no fundamental difference between irrationalities and (constructive) complex numbers: both are elements of algebraic extensions.

Quantum behavior in invariant subspaces of permutation representation. Illustration by A_5 acting on icosahedron

The alternating group A_5 is the smallest simple noncommutative group. It consists of 60 elements and its exponent is 30. Note that A_5 has a "physical incarnation": the fullerene C_{60} molecule has the structure of a Cayley graph of the group (see Fig. 1). This is clear from the following *presentation* of A_5 by two generators with three *relators*:

$$\mathsf{A}_5 \cong \left\langle a, b \mid a^5, b^2, (ab)^3 \right\rangle . \tag{2}$$

A_5 has five irreducible representations: the trivial $\mathbf{1}$ and four faithful $\mathbf{3}, \mathbf{3}', \mathbf{4}, \mathbf{5}$; and three *primitive* permutation representations having the following decompositions into the irreducible components: $\underline{\mathbf{5}} \cong \mathbf{1} \oplus \mathbf{4}$, $\underline{\mathbf{6}} \cong \mathbf{1} \oplus \mathbf{5}$, $\underline{\mathbf{10}} \cong \mathbf{1} \oplus \mathbf{4} \oplus \mathbf{5}$. Recall that a transitive action of a group on a set is called *primitive* [8], if there is no *non-trivial partition* of the set, invariant under the action of the group.

Consider the action of A_5 on the vertices Ω_{12} of an icosahedron. This action is transitive, but *imprimitive* with the non-trivial partition into the following blocks

$$\{|\ B_1 \mid \cdots \mid B_i \mid \cdots \mid B_6\ |\} \equiv \{|\ 1,7 \mid \cdots \mid i, i+6 \mid \cdots \mid 6,12\ |\}\,,$$

assuming the vertex numbering shown in Fig. 2. Each block B_i consists of a pair of opposite vertices of the icosahedron. Permutation representation of the action of A_5 on the icosahedron vertices has the following decomposition into irreducible components

$$\underline{\mathbf{12}} \cong \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{5} \quad \text{or} \quad \mathrm{T}^{-1}\,(\underline{\mathbf{12}})\,\mathrm{T} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{5}, \tag{3}$$

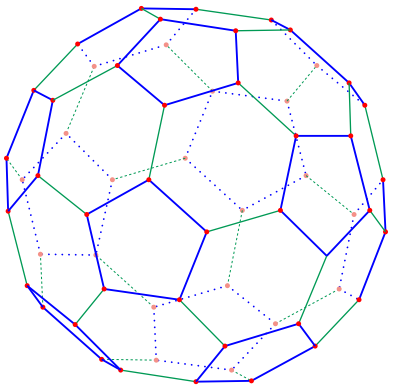


Figure 1: Cayley graph of A_5. Pentagons, hexagons and links between adjacent pentagons correspond to the relators a^5, $(ab)^3$ and b^2 in presentation (2).

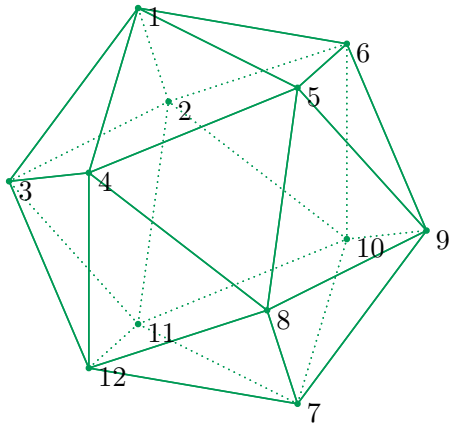


Figure 2: Icosahedron. Invariant blocks are pairs of opposite vertices.

where T is a matrix of transition from the "permutation" to "quantum" basis.

Actually there is no necessity to compute transformation matrices like T in (3) explicitly. There is a way [6] to express invariant scalar products in invariant subspaces in terms of easily computable matrices of *orbitals*, i.e., orbits of the action of G on the Cartesian product $\Omega \times \Omega$ (see e.g. [9, 10]).

In the case of action of A_5 on the set of icosahedron vertices Ω_{12}, the matrices of orbitals have the form

$$\mathcal{A}_1 = \mathrm{I}_{12},\ \mathcal{A}_2 = \begin{pmatrix} 0 & \mathrm{I}_6 \\ \mathrm{I}_6 & 0 \end{pmatrix},\ \mathcal{A}_3 = \begin{pmatrix} X & Y \\ Y & X \end{pmatrix},\ \mathcal{A}_4 = \begin{pmatrix} Y & X \\ X & Y \end{pmatrix}, \tag{4}$$

where I_n is $n \times n$ identity matrix, $X = \begin{pmatrix} \cdot & 1 & 1 & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & 1 & \cdot \end{pmatrix}$, $Y = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot \end{pmatrix}$. In terms of matrices (4) the invariant bilinear forms corresponding to decomposition (3) take the form

$$\begin{aligned} \mathcal{B}_{\mathbf{1}} =& \frac{1}{12}\left(\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4\right), \\ \mathcal{B}_{\mathbf{3}} =& \frac{1}{4}\left(\mathcal{A}_1 - \mathcal{A}_2 - \frac{1 + 2\mathsf{r}^2 + 2\mathsf{r}^3}{5}\mathcal{A}_3 + \frac{1 + 2\mathsf{r}^2 + 2\mathsf{r}^3}{5}\mathcal{A}_4\right), \end{aligned}$$

$$\mathcal{B}_{\mathbf{3}'} = \frac{1}{4}\left(\mathcal{A}_1 - \mathcal{A}_2 + \frac{1+2\mathsf{r}^2+2\mathsf{r}^3}{5}\mathcal{A}_3 - \frac{1+2\mathsf{r}^2+2\mathsf{r}^3}{5}\mathcal{A}_4\right),$$
$$\mathcal{B}_{\mathbf{5}} = \frac{5}{12}\left(\mathcal{A}_1 + \mathcal{A}_2 - \frac{1}{5}\mathcal{A}_3 - \frac{1}{5}\mathcal{A}_4\right),$$

where r is a 5th primitive root of unity. Note, that the cyclotomic integer $1+2\mathsf{r}^2+2\mathsf{r}^3$ is equal to $-\sqrt{5}$ — in fact, square root of any integer is a cyclotomic integer.

Now let us consider the scalar products of projections of "natural" vectors. If projections of vectors with natural components $m = (m_1, \ldots, m_{12})^T$ and $n = (n_1, \ldots, n_{12})^T$ onto the invariant subspaces corresponding to $\alpha = \mathbf{1}, \mathbf{3}, \mathbf{3}', \mathbf{5}$ are Φ_α and Ψ_α, respectively, then $\langle \Phi_\alpha \,|\, \Psi_\alpha \rangle = \langle m \,|\mathcal{B}_\alpha|\, n \rangle$. That is, we have

$$\langle \Phi_{\mathbf{1}} \mid \Psi_{\mathbf{1}} \rangle = \frac{1}{12}\Big(\langle m \,|\mathcal{A}_1|\, n\rangle + \langle m \,|\mathcal{A}_2|\, n\rangle + \langle m \,|\mathcal{A}_3|\, n\rangle + \langle m \,|\mathcal{A}_4|\, n\rangle\Big), \tag{5}$$

$$\langle \Phi_{\mathbf{3}} \mid \Psi_{\mathbf{3}} \rangle = \frac{1}{4}\left(\langle m \,|\mathcal{A}_1|\, n\rangle - \langle m \,|\mathcal{A}_2|\, n\rangle + \frac{\sqrt{5}}{5}\Big(\langle m \,|\mathcal{A}_3|\, n\rangle - \langle m \,|\mathcal{A}_4|\, n\rangle\Big)\right), \tag{6}$$

$$\langle \Phi_{\mathbf{3}'} \mid \Psi_{\mathbf{3}'} \rangle = \frac{1}{4}\left(\langle m \,|\mathcal{A}_1|\, n\rangle - \langle m \,|\mathcal{A}_2|\, n\rangle - \frac{\sqrt{5}}{5}\Big(\langle m \,|\mathcal{A}_3|\, n\rangle + \langle m \,|\mathcal{A}_4|\, n\rangle\Big)\right), \tag{7}$$

$$\langle \Phi_{\mathbf{5}} \mid \Psi_{\mathbf{5}} \rangle = \frac{5}{12}\left(\langle m \,|\mathcal{A}_1|\, n\rangle + \langle m \,|\mathcal{A}_2|\, n\rangle - \frac{1}{5}\Big(\langle m \,|\mathcal{A}_3|\, n\rangle + \langle m \,|\mathcal{A}_4|\, n\rangle\Big)\right). \tag{8}$$

Let us give some remarks on these expressions:

- Scalar product (5) can be written as

$$\langle \Phi_{\mathbf{1}} \mid \Psi_{\mathbf{1}} \rangle = \frac{1}{12}\,(m_1 + m_2 + \cdots + m_{12})\,(n_1 + n_2 + \cdots + n_{12})\,.$$

  In fact, generally, the trivial one-dimensional subrepresentation, contained in any permutation representation, can be interpreted as the "counter of particles", since the linear permutation invariant $\sum_{i=1}^{\mathsf{N}} n_i$ corresponding to this subrepresentation is the total number of elements of the set Ω in the ensemble.

- Expressions (6), (7) and (8) can take zero values on non-zero vectors with natural components. Thus we can observe a non-trivial "quantum behavior" — destructive interference — in the corresponding invariant subspaces.

- The Born probabilities for subrepresentations $\mathbf{3}$ and $\mathbf{3}'$ contain irrationalities that contradicts the frequency interpretation of probability for finite sets. Obviously, this is a consequence of the imprimitivity: one can not move an icosahedron vertex without simultaneous movement of its opposite. To resolve the contradiction, mutually conjugate subrepresentations $\mathbf{3}$ and $\mathbf{3}'$ must be considered together. The scalar product

$$\langle \Phi_{\mathbf{3}\oplus\mathbf{3}'} \mid \Psi_{\mathbf{3}\oplus\mathbf{3}'} \rangle = \frac{1}{2}\Big(\langle m \,|\mathcal{A}_1|\, n\rangle - \langle m \,|\mathcal{A}_2|\, n\rangle\Big)$$

  in the six-dimensional subrepresentation $\mathbf{3} \oplus \mathbf{3}'$ always gives rational Born's probabilities for vectors of natural "occupation numbers".

Conclusions

The universality of quantum mechanics, i.e., its applicability to physical objects of very different nature in a wide range of scales, suggests that a simple *a priori* mathematical scheme may lie in its basis. The fact that quantum behavior is demonstrated only by systems containing indistinguishable particles — any deviation from the exact identity of particles destroys quantum interferences — suggests that here the basic mathematical principle is symmetry. (Indistinguishability of objects means their belonging to the same orbit of the relevant symmetry group.) From this point of view, the quantum behavior is explained by the fundamental impossibility to trace individual objects in the process of evolution of ensembles of indistinguishable objects. The only, that is available from observations, is information about invariant combinations of such objects.

Our approach is based on the idea that any problem that has a meaningful empirical content, can be formulated in constructive, more definitely, finite terms.

Using general mathematical arguments we show that any quantum problem can be reduced to *permutations*. *Quantum interferences* are phenomena observed in invariant subspaces of permutation representations and expressed in terms of *permutation invariants*. If we assume also that the entities, which are subject to the permutations, have a physical meaning and that quantum amplitudes are vectors of multiplicities (which are natural numbers) of these entities in ensembles, we come to a very simple and self-consistent picture of the quantum behavior. The idea of natural quantum amplitudes looks very attractive. In particular, it allows to "deduce" the complex numbers which are postulated in standard quantum mechanics. If the idea is correct, then the quantum phenomena in different invariant subspaces are different manifestations — visible in appropriate "*observational setups*" — of a single process of permutations of the same collection of objects.

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Physics of Locally Non-Inertial Reference Frames

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Abstract: *The dynamics of a test body being a part of an extended mechanical system is considered. The components of the mechanical system may be in chaotic motion. Then the reference frame associated with this test body is a locally non-inertial reference frame. We consider the classical dynamics of the body in locally non-inertial reference frames.*

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Introduction

Consider an extended mechanical system comprising N particles with different masses, where N may tend to infinity. As a reference body, let us consider the center of mass of the mechanical system that can be associated with an inertial reference frame with the center of mass defined by the rule

$$X = \frac{\int_{-\infty}^{+\infty} x(t)\rho(x,t)dxdt}{\int_{-\infty}^{+\infty} \rho(x,t)dxdt} = \frac{\int_{-\infty}^{+\infty} \psi^*(x,t)x(t)\psi(x,t)dxdt}{\int_{-\infty}^{+\infty} \psi^*(x,t)\psi(x,t)dxdt},$$

ψ being the auxiliary function yielding squared the density of matter distribution $\rho(x,t)$. The test body (the observer), being a part of the extended system with chaotically moving components, is also observed moving chaotically. If the test body (the observer) is associated with the reference frame, the latter frame is a vibrating (i.e. non-inertial) one. Let us call the latter one a local non-inertial reference frame.

If we consider the trajectory of the test body as a smooth function possessing not only the first and the second time derivatives but as well the higher ones, then the transformation of any coordinate of the inertial reference frame associated with the center of mass of the mechanical system into a locally non-inertial reference frame shall be expresses as a converging Taylor series

$$X = x_0 + \dot{x}t + \frac{1}{2}\ddot{x}t^2 + \frac{1}{3!}\dddot{x}t^3 + ... + \frac{1}{k!}\dot{x}^{(k)}t^k + ...$$

Attempts to complement quantum mechanics with certain hidden variables, aimed at an opportunity to infer it from the classical mechanics, have been undertaken beginning from the first years of quantum mechanics.

Today those attempts look rather strange. In fact, in order to describe a mechanical system, the classical mechanics uses the finite number of spatial coordinates and their first and second order time derivatives, i.e. velocity and acceleration. Contrary to the classical mechanics, quantum mechanics may make use of infinite number of variables of the Hilbert space. Therefore introduction of additional hidden variables into the quantum mechanics seems unnecessary. Instead, it is reasonable to complement the classical mechanics with the infinite-dimensional space of higher-order derivatives of the spatial coordinates in order to integrate the classical and quantum mechanics into a unified theory with the common basis.

Why the Newton's second law is a second order differential equation?

The classical description of the physical reality contains an incomparably fewer number of variables than the quantum one. This invites the question: "How the classical description can be complemented?" While a possibility of augmenting the quantum mechanical description with additional ("hidden") variables has been

debated for long, the question as to how to complement the classical description to make it compatible with the quantum mechanical one has not received due attention.

Aristotelian physics assumed the velocity to be proportional to the applied force; hence the body dynamics shall be described by a first-order derivative differential equation. The Classical Physics in the inertial frames of reference postulates that a body not subject to any interactions maintains a constant velocity of the translational motion. Under these conditions, the body dynamics is described by a second-order differential equation, with acceleration being proportional to the force; this pertains to frames of reference called inertial ones. In this axiomatic, the dynamics of the body is described by a second-order differential equation with the acceleration being proportional to the force.

Let us consider the case of high-order differential equations.

Classical dynamics of test particle motion with higher-order time derivatives $\dot{q}^{(n)}$ of coordinates q was first described in 1850 by
M.Ostrogradsky [1] and is known as High-Order Derivative Ostrogradsky's Formalism. Being a mathematician, M. Ostrogradsky considered coordinate systems rather than frames of reference. This is just the case corresponding to a real reference frame comprising both inertial and non-inertial reference frames. In a general case, the Lagrangian takes on the form $(n \to \infty)$

$$L = L(q, \dot{q}, \ddot{q}, ..., \dot{q}^{(n)}). \tag{1}$$

Considering the dynamics of a body with an observer in a real reference frame, and varying the action function, we obtain the equation:

$$\delta S = \delta \int L(q, \dot{q}, \ddot{q}, ..., \dot{q}^{(n)})dt = \int \sum_{n=0}^{N} (-1)^n \frac{d^n}{dt^n} \frac{\partial L}{\partial \dot{q}^{(n)}} \delta \dot{q}^{(n)} dt = 0. \tag{2}$$

Applying the least action principle, we obtain Euler-Lagrange equation for the Extended Mechanics:

$$\sum_{n=0}^{N} (-1)^n \frac{d^n}{dt^n} \frac{\partial L}{\partial \dot{q}^{(n)}} = 0, \tag{3}$$

or

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} - ... + (-1)^N \frac{d^N}{dt^N} \frac{\partial L}{\partial \dot{q}^{(N)}} = 0. \tag{4}$$

Basic Postulates

Kinematic postulate

A free body maintains the same order of its coordinate time derivatives as the constant derivative involved in the transformation of the observer's reference frame to the relative of the inertial reference frame.

How could this classical/quantum description of the physical reality be extended with non-local inertial characteristics of vibrating reference frames? Is it better to complement the classical mechanics for better compatibility with the quantum mechanics than to do this with the quantum mechanics?

Dynamic postulate

The extended law of dynamics in real frame of references including the case of the vibration non-inertial frame

$$F = k_0 q + k_1 \dot{q} + k_2 \ddot{q} + k_3 \dddot{q} + ... + k_n \dot{q}^{(n)} + ...$$

When the classical mechanics is complemented with vibrating reference frames, a free particle turns into a randomly oscillating one with higher-order time derivatives of coordinates. Random oscillations of the observer's reference frame means a possibility of interference effects.

The classical/quantum constructions of Hybrid theory [2-6] as Neoclassical Physica could be based on the following common additional assumptions:

1. Any frame of reference is subject to random external influences. Hence, every reference frame is individual, and a transition from one reference frame to another may lead to a sudden change. The concept of the inertial frame in the classical mechanics is only valid on the average and, therefore, the Galilean relativity is an averaged concept as well. Hence, for any particle there exist several paths corresponding to different reference frames; the Heisenberg uncertainty can be interpreted as a consequence of the nonexistence of ideal inertial frames; and the Ehrenfest theorem can be considered a consequence of inertial frame being an averaged concept. Therefore, a free body preserves the same orders of its time derivatives as constant kinematic characteristics of the class of frame of references, e.g. in a uniformly accelerating reference frame a free body preserves its acceleration.

2. The Ehrenfest equations for the Quantum Mechanics observables are defined in coordinate averages. Compare the averaging procedure of this paper with the time averaging. Within the above framework, should the ideal inertial frames be non-existent, we could consider the averaging of the classical equations of motion over the time interval Δt:

$$-\frac{\partial U}{\partial q} = \frac{d}{dt}\frac{[p(t+\Delta t)+p(t-\Delta t)]}{2}.$$

Using the Taylor expansion

$$p(t \pm \Delta t) = p(t) \pm \dot{p}\Delta t + \frac{1}{2!}\ddot{p}(t)\Delta t^2 + ... + \frac{(-1)^n}{n!}\dot{p}^{(n)}(t)\Delta t^n + ..$$

the function $F = -\frac{\partial U}{\partial q}$ can be expanded as follows:

$$F(q.\dot{q}, \ddot{q}, \dddot{q}..., \dot{q}^{(k)}) = \dot{p}(t) + \frac{1}{2!}\dot{p}^{(3)}(t)\Delta t^2 + \frac{1}{4!}\dot{p}^{(5)}(t)\Delta t^4 + ...,$$

where $\dot{p}^{(n)}$ denotes n-th time derivative of momentum p. It is the Extended Law of Dynamics in a reality frame including the case of a vibrating non-inertial reference frame. Correspondingly, a free body preserves the same order of its time derivative as the constant kinematic characteristics of the reference frame possesses. For example, in a uniformly accelerating reference frame a free body preserves its acceleration.

3. The de-Broglie waves $\psi = \psi_0 \exp(-iS/\hbar)$ with the actions functions $S = S(q, \dot{q}, \ddot{q}, ..., \dot{q}^{(n)}, ...)$ can be considered as an embodiment of the fact that every reference frame is a vibrational one due to influences of random fields and waves, so that every free particle appears to be oscillating.

4. As the action function $S = S(q, \dot{q}, \ddot{q}, ..., \dot{q}^{(n)}, ...)$ is a convergent series in high derivatives of q the difference $\inf\left|S(q, \dot{q}, \ddot{q}, ..., \dot{q}^{(n)}, ...) - S(q, \dot{q})\right| = h$ is finite and can be identified with the constant h. Within the presented framework, the variables of the phase space (its high-order extension) provide an exhaustive description of the entire dynamics of a particle, but they cannot be measured because the ideal inertial frames of reference do not exist in reality. The infinite dimensionality of Hilbert space can also be understood as a consequence of all high order time derivatives being taken into account in the description of the dynamics [7-9].

Conclusion

From the very first steps of the quantum mechanics there have been numerous attempts to introduce there certain hidden variables providing the possibility of its seamless integration into the classical paradigm. Today it seems rather strange, as contrary to the classical mechanics, quantum mechanics may make use of infinite number of variables of the Hilbert space, and therefore, introduction of additional hidden variables is somehow superfluous here. At the same time, the classical mechanics uses the finite number of spatial coordinates and their first and second order time derivatives, i.e. velocity and acceleration. So, it is reasonable to complement the classical mechanics with the infinite-dimensional space of higher-order derivatives of the spatial coordinates in order to integrate the classical and quantum mechanics into a unified theory with the common basis.

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Exploiting correlations of continuous variable discordant states

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Abstract: *Correlations among quantum systems is a topic of key relevance in quantum information and quantum metrology since they offer the possibility to perform measurements of practical interest, expanding the measurement capability beyond the classical limits. Different quantities and strategies to distinguish whether correlations have a quantum nature or not have been introduced. Among optical states, Gaussian states represent a fundamental resource since they can be easily reproduced in laboratory. The most widespread Gaussian state is the thermal equilibrium state. When divided by a BS, the produced bipartite state shows correlations that can be exploited for imaging protocols and the results sometimes are comparable with quantum imaging protocols. Here we analyze ghost imaging application of both quantum and classical bipartite Gaussian states.*

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Introduction

The analysis and the application of the correlations among quantum systems is one of the major task of current research. Quantum correlations are the basis of the rapid development of quantum technologies and are becoming relevant in many other disciplines, ranging from biophysics [1–3] to cosmology [4]. These efforts have led to the introduction of different quantities and strategies to discriminate whether correlations have a quantum nature or not [1–5, 5]. Moreover, they are of the utmost importance in understanding the very foundations of quantum mechanics. Among all the possible states that exhibit quantum or classical correlations, very useful for practical applications is a thermal equilibrium state divided by a beam splitter (BS). This state is a continuous variable system state, that belongs to the family of Gaussian states, that are states with Gaussian Wigner functions [11–15]. In general, a thermal state is indeed a classical one according to the Glauber approach; however, the bipartite state emerging from the BS displays non-zero Gaussian discord and, thus, from the informational point of view it contains a non-vanishing amount of quantum correlations. Bipartite thermal equilibrium states are largely exploited in quantum optical labs because it is quite easy to generate and manipulate them. They are at the basis of most of the imaging protocols. A paradigmatic example of exploitation of correlations among those semiclassical states is the ghost imaging protocol, where the image of an object can be reconstructed from the correlations among the thermal beams exiting from the BS.

The ghost imaging protocol

In general, in ghost imaging protocols, one beam interacts with an object on one branch and then is detected by a detector without any spatial resolution (bucket detector), while the other branch (correlated to the previous one), which does not interact with the object, is registered by a spatially resolving detector, as, for example, an array of pixels. After K acquisitions, the image of the object is reconstructed by means of a correlation parameter $S(x_j)$, where x_j represents the position of the pixel j in the reference region where the average is performed experimentally over the K realizations. Hence, the idea at the basis of this protocol is that whether one disposes of two noise-correlated light beams, one crossing an object to be imaged and then detected by a bucket detector without any spatial resolution, the other addressed to a spatial resolving detector (as a CCD camera), then the image of the object can be reconstructed by considering correlations between the two measurements. Enlightened by the theoretical works of Klyshko [16], first time GI was experimentally realized

by Pittman et. al. in 1995 with two entangled beams produced by spontaneous parametric down-conversion (SPDC). Hence, this effect has been initially considered to be inherent to entanglement or in general quantum correlations. Later [17] there were realized ghost imaging in thermal fields, based on classical correlations of the spatial coordinates and momenta of thermal radiation. The classical correlations may be represented with the semi-classical theory only, but usage of a full quantum description provides comparison between the use of classical and quantum correlations in ghost imaging. In this work we consider different aspects of both classical and quantum ghost imaging. We made a comparison between four different correlation functions (CF) in terms of their performance in reconstructing the image of the object. The CFs that we have considered are listed below:

- the Glauber correlation function - $S(x_j) = G(2) = \langle N1N2(x_j)\rangle$;
- the normalized Glauber correlation function - $S(x_j) = g(2) = G(2)/\langle N1\rangle\langle N2(x_j)\rangle$;
- the covariance of the signals from two detectors (correlation functions of intensity fluctuations) - $S(x_j) = Cov(N1, N2(x_j)) \equiv \langle (N1 - \langle N1\rangle)(N2(x_j) - \langle N2(x_j)\rangle)\rangle$;
- the variance of the difference signal from two detectors - $S(x_j) = Var(N1 - N2(x_j)) \equiv \langle (N1 - N2(x_j) - \langle N1 - N2(x_j)\rangle)^2\rangle$;

where N1 and N2 are the number of the detected photons respectively in the bucket arm and in the other arm. If S is a general CF we can evaluate the quality of a ghost image using the signal to noise ratio (SNR) defined as

$$SNR \equiv \frac{\langle S_{max} - S_{min}\rangle}{\sqrt{Var(S_{max} - S_{min})}}, \tag{1}$$

where $Var(S_{max} - S_{min})$ is the variance of the image contrast.

Theoretical study

In our work, we have studied the influence of several parameters on the single-to-noise ratio: the brightness of the light source, related to the average number of photons μ per single spatial-temporal mode; the total number M of these modes collected by each element (pixel) of spatially resolving detector; the number of these elements, i.e. resolution of the image. The M number is the ratio of the detection volume to the coherence volume, so may be specified as $M = V_{det}/V_{coh} = (A_{pix} \cdot \tau_{det})/(A_{coh} \cdot \tau_{coh})$, where A_{pix} is the pixel size, τ_{det} - the detection (integration) time, A_{coh} - the coherence area (roughly, the speckle size, [18]), τ_{coh} - the coherence time of the light source. Usually in the experiment the detection time is much longer than the coherence time of the source $\tau_{det} \gg \tau_{coh}$.

From the point of view of photon statistics the total number of modes usually is $M \geq 1$, but for a very small detection volume ($A_{pix} \ll A_{coh}$ and $\tau_{det} \leq \tau_{coh}$) the total number of modes is equal to 1 ($M = 1$).

Because M is the number of modes collected by a pixel in a single acquired frame and μ is the number of photons per single mode, $\langle N_2\rangle = \eta_2 M\mu$ is the total number of photons detected in one pixel, where η is the quantum efficiency of the channel. The general efficiency of each channel $0 \leq \eta_i \leq 1 \quad (i = 1, 2)$ considers the transmission, collection and detection efficiency, i.e. the probability to detect an emitted photon. In our model the situation where a pixel is smaller than the coherence area ($A_{pix} \ll A_{coh}$) corresponds to a reduction of the collection efficiency.

The number of the spatial bits of the reconstructed image R depends on the relation between the pixel size and the coherence area size. When the pixel is equal or smaller than the size of a single spatial mode R is determined by the number of the spatial modes, otherwise it is given by the number of the pixels in the ghost image area. The optimal condition is $A_{pix} \simeq A_{coh}$ that maximizes the resolution of the reconstructed image and the collection efficiency. Concerning that in the experiment the size of the pixel of the CCD camera approximately matches the size of the speckle of the field. Actually in the case of quantum GI the pixel side is twice the coherence area, because in order to exploit the quantum correlation it was more important to have a high quantum efficiency η than a high resolution R.

For a single spatio-temporal mode the photon number (of detected photons) momenta can be calculated from the input-output relations of the process for PDC and pseudo-thermal light respectively. [19]

Results

The full analytical formulas are quite cumbersome, so we show only the ideal case of unity transmission/quantum efficiency and collection efficiency in both optical paths, $\eta_1 = \eta_2 = \eta = 1$ in table 2. Here and in the following the SNR values are normalized to the square root of the number of realizations $\sqrt{K}$. Let us discuss some specific regimes of the the performance of the ghost imaging in both pseudo-thermal and entangled light. Figure 1 represent the SNR with different correlation parameters as a function of illumination level for single-mode detection regime ($M = 1$) and ideal detection setup ($\eta = 1$).

	TwGI
$SNR_{G^{(2)}}$	$\frac{\sqrt{M\mu(1+\mu)}}{\sqrt{1+\mu(6+M+4MR)+\mu^2(6+M+6MR+2M^2R^2)}}$
$SNR_{g^{(2)}}$	$\frac{\sqrt{MR\mu(1+\mu)}}{\sqrt{1+\mu R(2+M+2MR)+\mu^2(-1+(3+M)R+2MR^2)}}$
SNR_{Cov}	$\frac{\sqrt{M\mu(1+\mu)}}{\sqrt{1+\mu(6+M+2MR)+\mu^2(6+M+2MR)}}$
SNR_{Var}	$\frac{\sqrt{2M\mu(1+\mu)}}{\sqrt{1+\mu(6+4MR)+\mu^2(6+4MR)}}$

	ThGI
$SNR_{G^{(2)}}$	$\frac{\sqrt{M}\mu}{\sqrt{1+2MR+2\mu(2+3MR+M^2R^2)+\mu^2(6+M+6MR+2M^2R^2)}}$
$SNR_{g^{(2)}}$	$\frac{\sqrt{MR}\mu}{\sqrt{-\mu(1+\mu)+(1+3\mu+(3+M)\mu 2)R+2M(1+\mu)2R^2}}$
SNR_{Cov}	$\frac{\sqrt{M}\mu}{\sqrt{1+2MR+4\mu(1+MR)+\mu^2(6+M+2MR)}}$
SNR_{Var}	$\frac{\sqrt{2M}\mu^{3/2}}{\sqrt{1+\mu(7+M(2+4R))+8\mu^2(1+MR)+\mu^3(6+4MR)}}$

Table 2: Expressions for the SNR of the reconstructed ghost image with thermal light (ThGI) and with twin beams (TwGI) for different protocols described in the text, in the lossless case ($\eta = 1$).

In the typical situation of large R ($R > 10$) it turns out that SNR for $g^{(2)}$, $Cov(N_1, N_2)$ and $Var(N_1 - N_2)$ behave the same for large values of detected photons $N \gg 1$, approaching approximatively the upper value of $(2R)^{-1/2}$, although they reach this bound at different values of the illumination level. For the pseudo-thermal light GI different protocols perform very similarly, the bound is reached at the high intensities of the detected light beams $N \gg 1$. Here $N = \eta M\mu$ is the "illumination", i.e. the total number of photons detected in the pixel in the single image. The performance of the scheme based on $G^{(2)}$ measurements is the worst for both quantum and thermal GI.

In case of twin-beams the SNR depends on the resolution, for the difference variance and covariance of two signals the SNR goes to plane for illumination level $N \gg 1/(2R)$, while our calculation shows that $g^{(2)}$ reaches the flat region as soon as $N \gg 1/(2R^2)$.

Concerning the performance of a very low-brightness source, Figure 1 shows that the advantage of using twin beams is in general very pronounced but, even with the same source, some protocol seems to be more convenient than others. In particular, for the pseudo-thermal light all the protocols scale as proportional to the number of detected photons per pixel N with the exception of the difference variance method, for which SNR approaches zero faster, i.e. proportional to $N^{3/2}$. For twin-beams GI all the methods lead to the same asymptotic behavior, $\propto N^{1/2}$. Therefore, we can conclude that for the GI of a complex object in case of classical or quantum illumination the three protocols exploiting $g^{(2)}$, $Cov(N_1, N_2)$ and $Var(N_1 - N_2)$ have the same performance at medium and high intensities, while for a low number photon fields, preference should be given to the normalized correlation function $g^{(2)}$.

For an ideal detection scheme ($\eta_1 = \eta_2 = 1$) the SNR of the ghost image in $G^{(2)}$ protocol drops down as $1/R$ ($R \gg 1$) and as $1/\sqrt{M}$ ($M \gg 1$) - see table 2. At at the same time, all the other protocols (exploiting $g^{(2)}$, $Cov(N_1, N_2)$ and $Var(N_1 - N_2)$) scale as $1/\sqrt{R}$ ($R \gg 1$) and asymptotically $\sim$ const with respect to M, for $M \gg 1$. That means the not-normalized and not-subtracted correlation function $G^{(2)}$ is not suitable for the GI, because of the worse imaging characteristics obtained with this protocol comparing with other parameters. Despite of that GI with $G^{(2)}$ is largely discussed in literature.

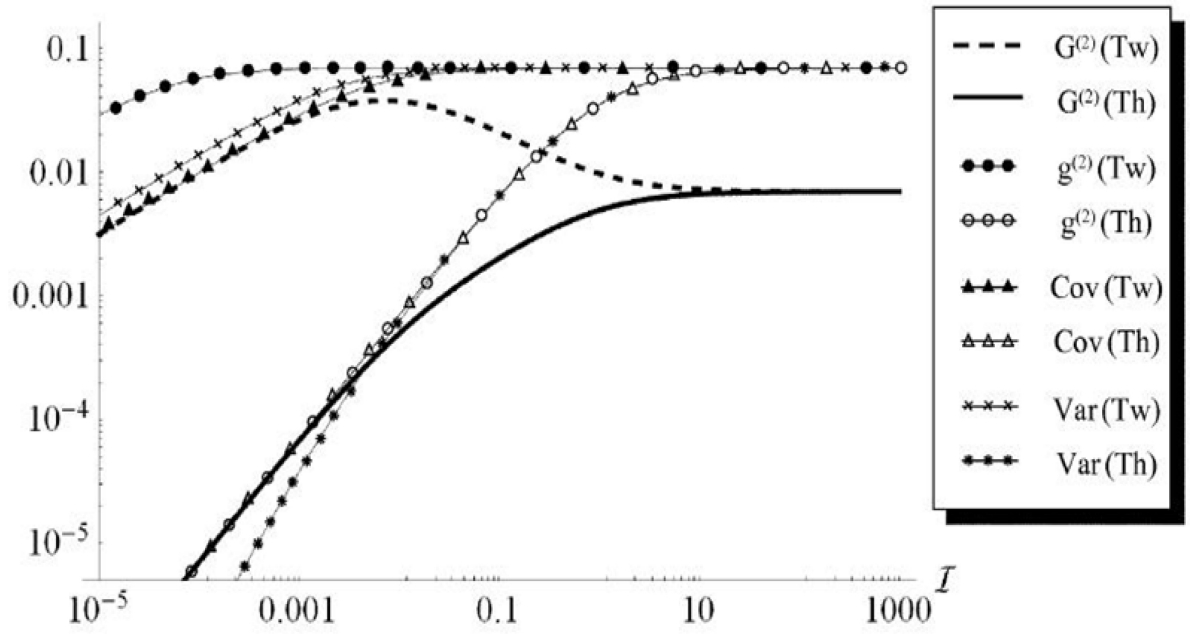

Figure 1: Dependence of the ghost image SNR on the average photon number per pixel for different protocols. The number of spatio-temporal modes per pixel M and the resolution parameter R are set, $M = 1$ and $R = 100$.

Conclusions

Our work demonstrated that correlations between semiclassical states, as thermal equilibrium state divided by a beam splitter, can be exploited in ghost imaging application with performances comparable with quantum correlations. This can be quantified using different correlation functions. Our theoretical model shows that for $g^{(2)}$, $Cov(N_1, N_2)$ and $Var(N_1 - N_2)$ GI with quantum light performs largely better than GI with semiclassical light at low illumination levels (few photons per pixel in a single shot), but they become equivalent at high illumination levels (many photons per pixel in a single shot).

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Soliton Like Structures as Traveling Nerve Pulses

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Abstract: *After some reduction procedure made on the nonlinear evolution equation for nerve pulses, based on thermodynamic principles, classic and non-classic traveling solutions have been obtained. We have studied these solutions for particular values in the parameter space, subjected to the nontrivial and condensate boundary conditions. We were able to obtain typical bell, and compacton like solutions for the first case and bubble and solitons on background for the second case. These nonlinear traveling waves could be responsible for transmitting efficiently the necessary information along the axons.*

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Introduction

As it is well known, the nature of the mechanisms for propagation signals along the nerve is one of the crucial problems in biophysics. Hodgkin and Huxley in the 50s [1] studied the complex dynamics of the ionic currents through voltage sensitive channels, and made detailed measurements of these currents. In 1966 Katz [2] proposed for traveling pulse the soliton traveling pulse as a simplest model for this activity. Now it is well established that the dynamic of ionic currents through voltage channels is responsible for the change of the membrane potential in nerve tissues.

Later, the Hodgkin and Huxley system (HH) was developed independently by Richard FitzHugh, and Jinichi Nagumo. Based on the work of Balthazar Van Der Pol, FitzHugh proposed a simplified neuronal model of Hodgkin and Huxley. For its part, Nagumo suggested as analogous neuronal, a nonlinear electrical circuit, controlled by an equation system also similar to those of Van Der Pol currents. The proposed simplified analog of these authors, is called FitzHugh-Nagumo model [3], [4]. By using an analytic technique, the homotopy analysis method (HAM), in the Fitzhugh–Nagumo equation, Abbasbandy [5], have found solitary wave solutions which are subjected to the control of new auxiliary parameter. Being susceptible to fairly complete analysis, the FHN system allows a qualitative understanding of the phenomenon of excitability, from the point of view of dynamical systems [43].

Recently, Heimburg and coworkers have developed a model for nerve pulses that support soliton like solutions [7, 8]. The model is constructed considering the nerve axon as a one dimensional cylinder with lateral density excitations moving along the axes. This model for the nerve pulses is based on the propagation of a localized density pulse (non linear wave) in the axon membrane. That is, resuming we can say that this theory is based on the lipid transition from a fluid to a gel phase at slightly below of body temperatures.

We suppose that along the axon, not only well famous bell solitons could propagate , but also non-classical soliton like solutions named compactons could be excited and travel carrying needed information. For the case of nontrivial boundary condition we will study the possibility for obtain bobtopogical solutions like bubbles and soliton on step. Thus, in the next section we briefly expose the main nonlinear evolution equation for nerve pulses. In the III section we show that compact solutions with the boundary trivial condition could appear and will travel with sonic, subsonic and supersonic velocities along the nerve. The section IV is devoted to discuss the appearance of bubbles and soliton on background, and finally in the last section we discuss some features around the found solutions and outline further implications of the model presented.

Equation of motion

The theory [7, 8] is based on hydrodynamic properties of a density pulse in the presence of dispersion. In the absence of dispersion and nonlinearity the equation of motion along the quasi-one dimensional axon is**:**

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial x}\left(c^2 \frac{\partial U}{\partial x}\right) \tag{1}$$

being $U = \rho^A - \rho_0^A$ the change of density in the membrane, ρ_0^A is the density of the membrane at physiological condition slightly above of melting transition.. Here c is the sound velocity with the value $c = (\kappa_s^A \rho^A)^{-1/2}$, being κ_s^A the compressibility. Subsequently, it is added the additional term responsible for dispersive effects $-h\frac{\partial^4 U}{\partial x^4}$. This term appears by experimental evidence on the relation between frequencies and velocities. For getting available nonlinear real dispersion relation the next term after the second derivative with respect to the special variable would stay the corresponding fourth derivative [8]. Next, it is assumed that the compressibility depends on the "field" U as a polynomial function [9]

$$c^2 = c_0^2 + pU + qU^2 + rU^3 + ... \tag{2}$$

Finally, the partial differential equation for difference of density is

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial x}\left\{\left(c_0^2 + pU + qU^2\right)\frac{\partial U}{\partial x}\right\} - h\frac{\partial^4 U}{\partial x^4} \tag{3}$$

The experiments made with nerve pulses shows the existence of narrow localized region of excitation. Thus, there is a big possibility that these pulses are in fact solitons but without tails. As is well known the classical solitons possess tails that could eventually interact at long distances that subsequently would degenerate the whole pulse to charge the needed information. So, the effort in this sense will be assumed to find solitons without tails i.e. solitary waves named regular or singular compactons.

Compactons and Classic Solutions

Let us investigate traveling (when $z = x - vt$) soliton solutions. Thus, the eq. (3) can be transformed to the next one

$$h\left(\tfrac{dU}{dz}\right)^2 = (c_0^2 - v^2)U^2 + \tfrac{p}{3}U^3 + \tfrac{q}{6}U^4 + 2CU + V_0 \tag{4}$$

with being p and q the parameters that appears in the Taylor expansion and V_0 the constant of integration.

By applying the trivial boundary condition

$$\text{if } z \to \infty, \text{ then } U \to 0 \text{ and } \frac{dU}{dz} \to 0 \tag{5}$$

produces $V_0 = 0$.

Solutions with sound velocity

First, let us consider the case: $c_0^2 = v^2$.

I. Sub-case $C = 0$. In this simple case by integrating the equation (4) we obtain the following solution

$$U(z) = \frac{2p}{\frac{p^2}{6}(z - z_0)^2 - q} \tag{6}$$

By avoiding singular behaviors, the possible solution in this case is achieved by assuming the negative value of the parameter q. The picture of this algebraic soliton solution is depicted in Figure1.

$$p = 3, q = -18\alpha$$

II. Sub-case :$C \neq 0$. For this sub-case we obtained the compact like soliton provided that

$$\frac{q^2}{p^3} = -\frac{8}{81C} \tag{7}$$

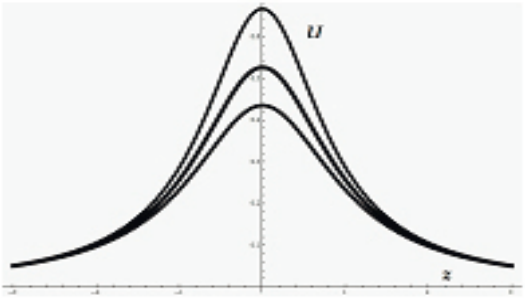

Figure 1: Algebraic soliton that travels with same velocity of sound along the axon

(12)

The integration of the equation (4) yields the following compact soliton like solution

$$U(\zeta) = \frac{Tan^2(\zeta)}{3B + 2BTan^2(\zeta)} \text{ for } \zeta \in [-\pi, \pi] \tag{8}$$

with the new variable $\zeta = \pm\frac{\sqrt{6BC}}{2}(z - z_0)$ and the parameter $B^2 = -\frac{p}{18C}$

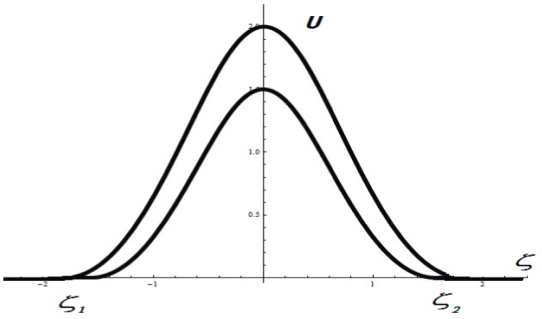

Figure 2: Compact soliton like structures that travels with the same velocity of sound along the axon.

Soliton solutions for which $v_0 \neq c_0$

Next, we investigate the solutions for which: $c_0^2 \neq v_0^2$ For this case, we slightly transform the right hand side of the equation (4)

$$h\left(\frac{dU}{dz}\right)^2 = CU(\kappa U + aU^2 + bU^3 + 1) \tag{9}$$

with the renamed parameters

$$\kappa = \frac{(c_0^2 - v^2)}{2C},\ a = \frac{p}{6C},\ b = \frac{q}{12C} \tag{10}$$

Without lost of generality we can choice $h = 1$. Then, after integration we again obtain a perfect compact like solution.

$$U = \frac{Tan^2(\xi)}{(3A - \kappa) + (2A - \kappa)\ Tan^2(\xi)},\quad \text{for} \quad |\ \xi\ | \leq 2\pi \tag{11}$$

with

$$\xi = \pm\tfrac{1}{2}\sqrt{C(3A - \kappa)}(z - z_0)$$

and $b = (\kappa - 2A)A^2,\ a = 2A\kappa - 3A^2$

As can be observed this specific solution exists by avoiding singularities i.e. in the particular case only for negative parameter values

$$B = \kappa - 2A < 0 \tag{12}$$

The plot of the equation 11 is similar to the figure of compact soliton represented in the Figure 1and shows the robust configuration of the soliton like solution. For these moving compactons with velocities differing from the sound one, the parameters a and b satisfy the following relation

$$b = \frac{1}{27}(k - 2\sqrt{\kappa^2 - 3a})(\kappa + \sqrt{\kappa^2 - 3a})^2 \tag{13}$$

This expression depends on the soliton velocity by virtue of equation (10). Outside of this dependence between the parameters the solutions probably will not admit compact support.

By deploying the constraint (12) we obtain the parameter region of available velocities. Thus, when $\kappa > 0$ the velocity corresponds to subsonic waves, i.e.

$$v^2 < c_0 - 2\sqrt{\frac{pC}{3}} \tag{14}$$

If the values of $\kappa < 0$ we obtain supersonic soliton waves with the next segment of values

$$c_0^2 < v^2 < c_0^2 + 2\sqrt{\frac{pC}{3}} \tag{15}$$

Energy of the obtained solitons

The energy of solutions obtained for the ordinary nonlinear differential equation i.e. for traveling solutions can be calculated by using the energy density, which in some $\xi_0 = 1.57, A = 1, k = 1, C = 2$

The integration with respect to the variable ξ in the interval $[-\pi, \pi]$ for compactons can be directly evaluated using the energy density. From the equation (4) one can notice that the kinetic energy is equal to the potential one for the "mechanical analog particle". This leads to calculate the total energy using the following equation

$$E = \int_{z_0}^{+z_0} \frac{W(U)}{h} dz \tag{16}$$

with the "potential" piece of energy density

$$W(u) = (c_0^2 - v^2)U^2 + \frac{p}{3}U^3 + \frac{q}{6}U^4 + 2CU. \tag{17}$$

This total energy depends on the velocity of the non-classical solution. For avoiding singular solutions, when the values $C \neq 0$, traveling compact solution could emerge only when the velocities satisfy the relation $2A - \kappa > 0$, that is in complete accordance with the previous analysis made on the parameter space (12) for obtaining subsonic and also supersonic compact solutions.

Condensate boundary condition.

Now let us study the equation (4) by keeping in mind first the non-trivial or the condensate boundary condition

$$\text{if } z \to \infty \text{ then } U \to U_0 \text{ and } \frac{dU}{dz} \to 0 \tag{18}$$

After subsequent integration one can obtain the next equation

$$\begin{aligned} &h\left(\tfrac{dU}{dz}\right)^2 = CU + (c_0^2 - v^2)(U)^2 + \alpha(U)^3 + \\ &\beta(U)^4 + V_0 \end{aligned} \tag{19}$$

Where the parameters V_0 and U_0 satisfy the next equation

$$V_0 = U_0[(v^2 - c_0^2)U_0 - \alpha U_0^2 - \beta U_0^3 - C] \tag{20}$$

Traveling sonic solution

As for the case of trivial boundary condition, let us consider the case: $c_0^2 = v^2$. In this case the equation (4) could be transformed

$$\pm\sqrt{\frac{\beta}{h}}(z - z_0) = \int \frac{1}{r\sqrt{r^2 + Gr + M}} dr \tag{21}$$

with

$$U = r - a,\ M = 6a^2 - 3\alpha' a \text{ and } G = \alpha' - 4a \tag{22}$$

and $\alpha' = \frac{\alpha}{\beta},\ V_0' = \frac{V_0}{\beta}$ where the parameter a needs to satisfy the algebraic cubic equation

$$a^3 - \left(\frac{3\alpha}{4\beta}\right) a^2 - \frac{C}{4\beta} = 0$$

Having obtained the value of parameter a from the cubic equation (6), it should be easy to calculate the value of parameter M by using equation (5). In order to integrate the equation (4) and obtain analytical and non singular solutions we must put the condition for the discriminant of the expression under the square in the Eq.(4) as follows

$$D = 4M - G^2 = 8a^2 - 4a\alpha' - \alpha'^2 < 0.$$

Under all these requirements we can make an assumption that regular localized soliton-like solutions exist when the parameters $a,\ \frac{\alpha}{\beta}$ satisfy whichever of these two inequalities

$$\begin{array}{l} I)\ a > \frac{\alpha}{4\beta}(1 + \sqrt{3}) \\ II)\ \frac{\alpha}{2\beta} < a < \frac{\alpha}{4\beta}(1 + \sqrt{3}) \end{array} \tag{23}$$

These parameter restrictions will be the conditions for the existence of a set of non-topological like solitons.

By inverting the integral (4) written above, one has finally the following solution by avoiding singular behavior

$$U(z) = \frac{2M}{\sqrt{-D}\ Cosh\left\{\sqrt{\frac{M\beta}{h}}(z - z_0)\right\} - G} - a \tag{24}$$

As usual, for qualitative purposes, this solution can be visualized by taking concrete parameter values. For instance, for a good picture presentation let us suppose that $\beta = 2$ and according to the work [10] for unilamellar DPPC vesicles we can take for example the value $\alpha = -7$. Thus, the other important parameters should estimated straightforward, and after reparameterization of variables finally the resulting picture is depicted in Figure 4.

Super and sub - sonic traveling non-topological solutions

Let us now investigate the other case when the values of the velocities of traveling structures are different than the sound one. We replace the value V_0 of eq. (19) in the equation (4) and obtain for $y = U - U_0$

$$h\left(\frac{dy}{dz}\right)^2 = y^2(A + By + \beta y^2) \tag{25}$$

This equation is obtained considering the following relations of the parameter values:

$$C = -[2U_0(c_o^2 - v^2) + 2\alpha U_o^2 + 4U_o^2 + 4U_o^3\beta] \tag{26}$$

$$A = (c_o^2 - v^2) + 3\alpha U_0 + 6\beta U_0^2 \tag{27}$$

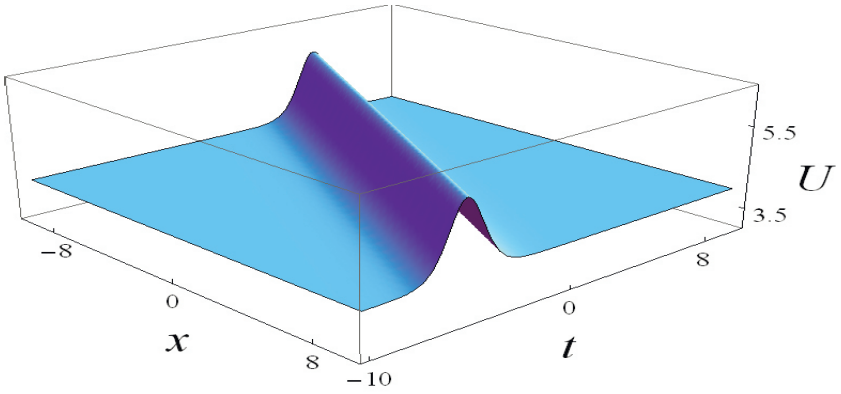

Figure 3: Typical "bright" soliton on background traveling with the same velocity of sound along the axon.

$$B = \alpha + 4\beta U_0 \tag{28}$$

To avoid singular behavior, let us suppose that parameters A, B and β satisfy the next inequalities

$$A > 0,\ \delta = 4A\beta - B^2 < 0 \tag{29}$$

Thus, after the corresponding integration one can obtain the next representation of the solution

$$U = \frac{\Gamma + U_0 Sinh^2\sqrt{A}(\xi - \xi_0)}{\sigma + Cosh^2(\sqrt{A}(\xi - \xi_0))} \tag{30}$$

provided that

$$\Gamma = \frac{A}{\sqrt{-\delta}} + \frac{U_0}{2}\left(\frac{m^2}{\sqrt{-\delta}} + 1\right),\ \sigma = \frac{1}{2}\left(\frac{m^2}{\sqrt{-\delta}} - 1\right)$$

and $z - z_0 = 2\sqrt{h}(\xi - \xi_0)$, $\quad B = -m^2$. The solution exist if the next restrictions holds for the soliton velocities $3\alpha U_0 + 6\beta U_0^2 > v^2 - c_0^2 > \alpha U_0 + 2\beta U_0^2 - \frac{\alpha^2}{4\beta}$

The bubble soliton can be easily seen when we choice appropriated parameter values. The simplest ones could be generated when the relation $\Gamma < \sigma$ in Equation (19) holds for determined parametric values.

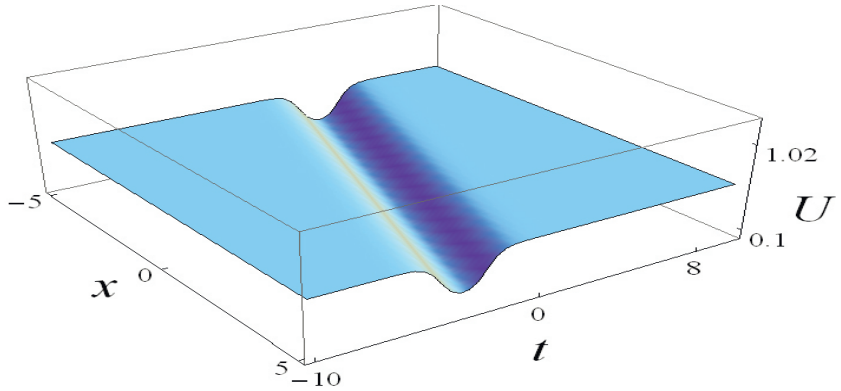

Figure 4: By using the reparameterization of variables in equation (30) one can depict the bubble like soliton on the background.

The bubble or in some sense the small dip or rarefaction of density and the soliton on the condensate are propagating with some velocity whose values are restricted by the equation (17). Along with the existence of bubble type of solitons, the soliton on the background also appears that should be dual to the first one.

Conclusions

We have discussed in this contribution the appearance of classical and non - classical soliton like pulses along the axon based on the pioneering work of [7,9,10]. The classical bell shape soliton shows infinite tails in contrast to the non classical solitons i.e. compactons without tails. Also we have obtained bubble and pedestal solitons for the case of condensate boundary conditions. For the case of trivial boundary condition these solutions could propagate with arbitrary velocities but restricted by parametric equations (12), i.e. sonic, super - and sub-sonic solutions could appear in this model. This means that these waves could charge the information efficiently faster between the two significant separated centers along the nerve axon. It should be interesting to check which one of these possibilities take place in real experiments. The properties (including its shape and its energy) can be determined uniquely as a function of compacton or algebraic soliton velocity. Given a measured nonlinear wave velocity, the theory contains freely adjustable parameters and has the virtue of being feasible. As soon as the nonlinear excitations in whichever segment of the axon structure is being activated by the presence of external agents for example, phonons, etc, compacton or bell solitons could appear, describing realistically the density displacements.

For the case of condensed boundary condition the traveling small dip or rarefaction and soliton excitation on the background can exist and can run with constant velocity along the nerve. Thus, the long pulse plateau in the nerve could be perturbed by bubble and bright solitons on the background. Therefore, in both directions of the axis, at long distances from the active zone, the density displacements will maintain their value, forming the non-vanishing boundary condition.

These solutions could eventually be responsible for various fundamental processes inside the nerve, especially those processes that involve some kind of parametric phase transitions. This is because of the realistic interpretation of bubbles as a nucleus of some stable phases in the bubble vacuum or a metastable one. Also, both solutions, that is, the bubble and the soliton on the background obtained here as particular soliton-like solutions for specific values of parameters, could be used by the nerve system for enhancing confidentiality in communication tasks. For instance, as the bubble soliton amplitude vanishes or minimizes during propagation along the nerve, this wave could be used to perform communication transmission for security, whereas the required information can be retrieved by the dark/bright soliton conversion on the background.

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New results on general description of Dirac particles in strong gravitational fields

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Abstract: *We discuss the dynamics of the Dirac fermions in the general strong gravitational and electromagnetic fields. We derive the general Hermitian Dirac Hamiltonian and transform it to the Foldy-Wouthuysen representation for the spatially isotropic metric. The quantum operator equations of motion are obtained and the semiclassical limit is analyzed. The comparison of the quantum mechanical and classical equations shows their complete agreement. The helicity dynamics in strong fields is discussed. Squaring the covariant Dirac equation explicitly shows a similarity of the interactions of electromagnetic and gravitational fields with a charged and spinning particle.*

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Introduction

The study of spin dynamics in a curved spacetime (a gravitational field) was initiated immediately after the formulation of the relativistic Dirac theory. The early efforts were mainly concerned with the development of mathematical tools and methods appropriate for the description of the interaction of spinning particles with the gravitational field. The studies of the spinor analysis in the framework of the general Lagrange-Noether approach have subsequently resulted in the construction of the gauge-theoretic models of physical interactions, including also gravity (see Refs. [1, 2] and references therein).

At a later stage, considerable attention was turned to the investigation of the specific physical effects in the gravitational field predicted for quantum, semiclassical, and classical relativistic particles with spin. Various aspects of the dynamics of fermions were studied in weak gravitational fields, i.e., for the case when the spacetime geometry does not significantly deviate from the flat Minkowski manifold. Another class of problems was the analysis of trajectories of semiclassical and classical particles in gravitational field configurations being exact solutions of Einstein's equations like the Schwarzschild and Kerr metrics. The behavior of the spin in strong gravitational fields represents another interesting subject with possible applications to the study of the physical processes in the vicinity of massive astrophysical objects and near black holes. For example, the overview of the important mathematical subtleties is given in Refs. [3–6].

In this paper, we present the results of our investigations of the Dirac fermions based on the new method [7] of the Foldy-Wouthuysen (FW) transformation. Earlier, we analyzed the dynamics of spin in weak static and stationary gravitational fields [8–11] and in strong stationary gravitational fields [12] of massive compact sources. These previous results are now extended on the general case of a completely arbitrary gravitational field.

The paper is organized as follows. In Sec. , we give preliminaries for the description of the general metric and the coframe, and then derive the Hermitian Dirac Hamiltonian in an arbitrary curved spacetime. For completeness, we also consider the electromagnetic interaction. In Sec. , we outline the FW technique and derive the FW Hamiltonian together with the corresponding operator equations of motion. The central result is

the derivation of the equation of spin precession an arbitrary gravitational field. The quantum and semiclassical spin dynamics is compared with the dynamics of a classical spin in Sec. . We use the standard formalism of Mathisson and Papapetrou, and discuss the Hamiltonian approach. The results obtained are summarized in Sec. .

Our notations and conventions are the same as in Ref. [9]. In particular, the world indices of the tensorial objects are denoted by Latin letters $i, j, k, \ldots = 0, 1, 2, 3$ and the first letters of the Greek alphabet label the tetrad indices, $\alpha, \beta, \ldots = 0, 1, 2, 3$. Spatial indices of 3-dimensional objects are denoted by Latin letters from the beginning of the alphabet, $a, b, c, \ldots = 1, 2, 3$. Temporal and spatial tetrad indices are distinguished by hats.

Dirac particle in a gravitational field

General parametrization of the spacetime metric

Let us recall some basic facts and introduce the notions and objects related to the description of the motion of a classical spinning particle in a curved manifold. The massive particle is quite generally characterized by its position in spacetime, $x^i(\tau)$, where the local spacetime coordinates are considered as functions of the proper time τ, and by the tensor of spin $S^{\alpha\beta} = -S^{\beta\alpha}$. The analysis of the dynamics of the classical spinning particle is given later in Sec. .

We denote 4-velocity of a particle $U^\alpha = e_i^\alpha dx^i/d\tau$. In view of the choice of parametrization by the proper time, it is normalized by the standard condition $g_{\alpha\beta}U^\alpha U^\beta = c^2$ where $g_{\alpha\beta} = \mathrm{diag}(c^2, -1, -1, -1)$ is the flat Minkowski metric. We use the tetrad e_i^α (or coframe) to describe the dynamics of spinning particles on a spacetime manifold in arbitrary curvilinear coordinates. When the spacetime is flat and the gravitational field is absent, one can choose the global Cartesian coordinates and the holonomic orthonormal frame coinciding with the natural one, $e_i^\alpha = \delta_i^\alpha$. The spacetime metric is related to the coframe field in the usual way: $g_{\alpha\beta}e_i^\alpha e_i^\beta = g_{ij}$.

We use the notations t and x^a $(a = 1, 2, 3)$ for the coordinate time and the spatial local coordinates, respectively. There are many different ways to represent a general spacetime metric. A convenient parametrization of the spacetime metric was proposed by De Witt [13] in the context of the canonical formulation of the quantum gravity theory. In a slightly different disguise, the general form of the line element of an arbitrary gravitational field reads

$$ds^2 = V^2c^2dt^2 - \delta_{\hat{a}\hat{b}}W^{\hat{a}}{}_c W^{\hat{b}}{}_d \,(dx^c - K^c cdt)\,(dx^d - K^d cdt). \tag{1}$$

This parametrization involves more functions than the actual number of the metric components. Indeed, the total number of the functions $V(t, x^c)$, $K^a(t, x^c)$, and $W^{\hat{a}}{}_b(t, x^c)$ is $1 + 3 + 9 = 13$ which is greater than 10. However, we have to take into account that the line element (1) is invariant under redefinitions $W^{\hat{a}}{}_b \longrightarrow L^{\hat{a}}{}_{\hat{c}}W^{\hat{c}}{}_b$ using arbitrary local rotations $L^{\hat{a}}{}_{\hat{c}}(t, x) \in SO(3)$. Subtracting the 3 rotation degrees of freedom, we recover exactly 10 independent variables that describe the general metric of the spacetime.

Dirac equation

One needs the orthonormal frames to discuss the spinor field and to formulate the Dirac equation. From the mathematical point of view, the tetrad is necessary to "attach" a spinor space at every point of the spacetime manifold. Tetrads (coframes) are naturally defined up to a local Lorentz transformations, and one usually fixes this freedom by choosing a gauge. We discussed the choice of the tetrad gauge in [11] and have demonstrated that a physically preferable option is the Schwinger gauge [14, 15], namely the condition $e_a^{\hat{0}} = 0, a = 1, 2, 3$. Accordingly, for the general metric (1) we will work with the tetrad

$$e_i^{\hat{0}} = V\,\delta_i^0, \quad e_i^{\hat{a}} = W^{\hat{a}}{}_b\left(\delta_i^b - cK^b\,\delta_i^0\right), \quad a = 1, 2, 3. \tag{2}$$

The inverse tetrad, such that $e_\alpha^i e_j^\alpha = \delta_j^i$,

$$e_{\hat{0}}^i = \frac{1}{V}\left(\delta_0^i + \delta_a^i cK^a\right), \quad e_{\hat{a}}^i = \delta_b^i W^b{}_{\hat{a}}, \quad a = 1, 2, 3, \tag{3}$$

also satisfies the similar Schwinger condition, $e_{\hat{a}}^0 = 0$. Here we introduced the inverse 3×3 matrix, $W^a{}_{\hat{c}}W^{\hat{c}}{}_b = \delta_b^a$.

The following observation will be useful for the subsequent computations. A classical massive particle moves along a world line $x^i(\tau), i = 0,1,2,3$, parametrized by the proper time τ. Its 4-velocity is defined as usual by the derivatives $U^i = dx/d\tau$. With respect to a given orthonormal frame, the velocity has the components $U^\alpha = e^\alpha_i U^i, \alpha = 0,1,2,3$. It is convenient to describe the 4-velocity by its 3 spatial components $v^{\widehat{a}}, a = 1,2,3$, in an anholonomic frame. Then $U^\alpha = (\gamma, \gamma v^{\widehat{a}})$, with the Lorentz factor $\gamma^{-1} = \sqrt{1 - v^2/c^2}$, and, consequently,

$$U^0 = \frac{dt}{d\tau} = e^0_\alpha U^\alpha = \frac{\gamma}{V}, \tag{4}$$

$$U^a = \frac{dx^a}{d\tau} = e^a_\alpha U^\alpha = \frac{\gamma}{V}\left(cK^a + VW^a{}_{\widehat{b}}\, v^{\widehat{b}}\right). \tag{5}$$

We used (3) here. Dividing (5) by (4) and denoting

$$\mathcal{F}^a{}_b = VW^a{}_{\widehat{b}},$$

we find for the velocity with respect to the coordinate time

$$\frac{dx^a}{dt} = \mathcal{F}^a{}_b\, v^b + cK^a. \tag{6}$$

The Dirac equation in a curved spacetime reads

$$(i\hbar\gamma^\alpha D_\alpha - mc)\Psi = 0, \qquad \alpha = 0,1,2,3. \tag{7}$$

This equation is invariant under the general transformations of the spacetime coordinates (under diffeomorphism), and is covariant under the local Lorentz transformations. Recall that the Dirac matrices γ^α are defined in local Lorentz (tetrad) frames and they have constant components. The spinor covariant derivatives are consistently defined in the gauge-theoretic approach [1,2,16,17] as

$$D_\alpha = e^i_\alpha D_i, \qquad D_i = \partial_i + \frac{iq}{\hbar} A_i + \frac{i}{4}\sigma^{\alpha\beta}\Gamma_{i\,\alpha\beta}. \tag{8}$$

Here the Lorentz connection is $\Gamma_i{}^{\alpha\beta} = -\Gamma_i{}^{\beta\alpha}$, and $\sigma^{\alpha\beta} = \frac{i}{2}\left(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right)$ are the generators of the local Lorentz transformations of the spinor field. For completeness, we assumed that the Dirac particle is charged and the electric charge q describes its coupling to the 4-potential of the electromagnetic field A_i. Note that the canonical dimension of the electromagnetic field strength 2-form $F = dA$ and of the electromagnetic 1-form $A = A_i dx^i$ is $[F] = [A] = [\hbar/q]$, see [18]. The gravitational and inertial effects (which are deeply related to each other in the framework of the gauge-theoretic approach) are encoded in coframe and connection in (7),(8); for the relevant discussion see Refs. [19–21] and references therein.

Using the parametrization of the general metric (1) with the tetrad (2) in the Schwinger gauge, we find explicitly the components of connection

$$\Gamma_{i\,\widehat{a}\widehat{0}} = \tfrac{c^2}{V} W^b{}_{\widehat{a}}\, \partial_b V\, e_i{}^{\widehat{0}} - \tfrac{c}{V}\, \mathcal{Q}_{(\widehat{a}\widehat{b})}\, e_i{}^{\widehat{b}}, \tag{9}$$

$$\Gamma_{i\,\widehat{a}\widehat{b}} = \tfrac{c}{V}\, \mathcal{Q}_{[\widehat{a}\widehat{b}]}\, e_i{}^{\widehat{0}} + \left(\mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}} + \mathcal{C}_{\widehat{a}\widehat{c}\widehat{b}} + \mathcal{C}_{\widehat{c}\widehat{b}\widehat{a}}\right) e_i{}^{\widehat{c}}. \tag{10}$$

Here we introduced the two useful objects:

$$\mathcal{Q}_{\widehat{a}\widehat{b}} = g_{\widehat{a}\widehat{c}} W^d{}_{\widehat{b}}\left(\tfrac{1}{c}\dot{W}^{\widehat{c}}{}_d + K^e \partial_e W^{\widehat{c}}{}_d + W^{\widehat{c}}{}_e \partial_d K^e\right), \tag{11}$$

$$\mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{c}} = W^d{}_{\widehat{a}} W^e{}_{\widehat{b}}\, \partial_{[d} W^{\widehat{c}}{}_{e]}, \quad \mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}} = g_{\widehat{c}\widehat{d}} \mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{d}}. \tag{12}$$

As usual, the dot $\dot{}$ denotes the partial derivative with respect to the coordinate time t. One can immediately recognize that $\mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{c}} = -\mathcal{C}_{\widehat{b}\widehat{a}}{}^{\widehat{c}}$ is the anholonomity object for the spatial triad $W^{\widehat{a}}{}_b$. The indices (that all run

from 1 to 3) are raised and lowered with the help of the spatial part of the flat Minkowski metric, $g_{\hat{a}\hat{b}} = -\delta_{ab} = \mathrm{diag}(-1,-1,-1)$.

One can derive the Dirac equation from the action integral

$$I = \int d^4x\,\sqrt{-g}\,L,$$

with the Lagrangian (recall for the conjugate spinor $\overline{\Psi} := \Psi^\dagger\gamma^{\hat{0}}$)

$$L = \frac{i\hbar}{2}\left(\overline{\Psi}\gamma^\alpha D_\alpha\Psi - D_\alpha\overline{\Psi}\gamma^\alpha\Psi\right) - mc\,\overline{\Psi}\Psi. \tag{13}$$

A direct check shows that, with (8)-(12) inserted, the Schrödinger form of the Dirac equation derived from this action involves a non-Hermitian Hamiltonian. However, this problem is fixed if we introduce the new wave function by [12]

$$\psi = \left(\sqrt{-g}e^0_{\hat{0}}\right)^{\frac{1}{2}}\,\Psi. \tag{14}$$

Such a non-unitary transformation also appears in the framework of the pseudo-Hermitian quantum mechanics [22,23] (cf. [24]).

Variation of the action with respect to the *rescaled* wave function ψ yields the Dirac equation in Schrödinger form $i\hbar\frac{\partial\psi}{\partial t} = \mathcal{H}\psi$. The corresponding *Hermitian* Hamiltonian reads [12]

$$\begin{array}{l}\mathcal{H} = \beta mc^2 V + q\Phi + \frac{c}{2}\left(\pi_b\mathcal{F}^b{}_a\alpha^a + \alpha^a\mathcal{F}^b{}_a\pi_b\right)\\ \quad + \frac{c}{2}\left(\boldsymbol{K}\cdot\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\boldsymbol{K}\right) + \frac{\hbar c}{4}\left(\boldsymbol{\Xi}\cdot\boldsymbol{\Sigma} - \Upsilon\gamma_5\right).\end{array} \tag{15}$$

Here $\boldsymbol{K} = \{K^a\}$, and the kinetic momentum operator $\boldsymbol{\pi} = \{\pi_a\}$ with $\pi_a = -i\hbar\partial_a + qA_a = p_a + qA_a$ accounts for the interaction with the electromagnetic field $A_i = (\Phi, A_a)$. To remind the notation: $\beta = \gamma^{\hat{0}}, \boldsymbol{\alpha} = \{\alpha^a\}, \boldsymbol{\Sigma} = \{\Sigma^a\}$, where the 3-vector-valued Dirac matrices have their usual form; namely, $\alpha^a = \gamma^{\hat{0}}\gamma^a$ $(a,b,c,\cdots = 1,2,3)$ and $\Sigma^1 = i\gamma^{\hat{2}}\gamma^{\hat{3}}, \Sigma^2 = i\gamma^{\hat{3}}\gamma^{\hat{1}}, \Sigma^3 = i\gamma^{\hat{1}}\gamma^{\hat{2}}$. We also introduced a pseudoscalar Υ and a 3-vector $\boldsymbol{\Xi} = \{\Xi_a\}$ by [12]

$$\begin{array}{l}\Upsilon = V\epsilon^{\hat{a}\hat{b}\hat{c}}\Gamma_{\hat{a}\hat{b}\hat{c}} = -V\epsilon^{\hat{a}\hat{b}\hat{c}}\mathcal{C}_{\hat{a}\hat{b}\hat{c}},\\ \Xi_{\hat{a}} = \frac{V}{c}\,\epsilon_{\hat{a}\hat{b}\hat{c}}\Gamma_{\hat{0}}{}^{\hat{b}\hat{c}} = \epsilon_{\hat{a}\hat{b}\hat{c}}\,\mathcal{Q}^{\hat{b}\hat{c}}.\end{array} \tag{16}$$

Note that we have fixed a number of small points with the signs and numeric factors, and one should be careful when comparing the above formulas with the earlier results presented in [12]. For the static and stationary rotating configurations, the pseudoscalar invariant vanishes ($\epsilon^{\hat{a}\hat{b}\hat{c}}\mathcal{C}_{\hat{a}\hat{b}\hat{c}} = 0$), and thus the corresponding term was absent in the special cases considered in Refs. [11,12]. But in general this term contributes to the Dirac Hamiltonian.

It is worthwhile to mention that the recent discussion [25] of the Dirac fermions in an arbitrary gravitational field is very different. In deep contrast to the explicitly Hermitian Hamiltonian (15), the non-Hermitian one is used In Ref. [25].

The Foldy-Wouthuysen transformation

In the previous section, we described the dynamics of the fermion in the Dirac representation. The physical contents of the theory is however revealed in the FW representation [26]. We will now construct the FW Hamiltonian for the fermion moving in an arbitrary gravitational field described by the general metric (1). We start with the exact Dirac Hamiltonian (15) and apply the method developed in [7].

Just like before in our earlier works [11,12], we do not make any assumptions and/or approximations for the functions $V, W^{\hat{a}}{}_b, K^a$. The Planck constant $\hbar$ will be treated as the only small parameter. In accordance with this strategy, we retain in the FW Hamiltonian all the terms of the zero and first orders in $\hbar$. The leading nonvanishing terms of order $\hbar^2$ have been calculated in both nonrelativistic and weak field approximations in our previous works [8,11,12] for the more special cases. These terms describe the gravitational contact (Darwin) interaction.

General preliminaries

A generic Hamiltonian can be decomposed into operators that commute and anticommute with β:

$$\mathcal{H}=\beta\mathcal{M}+\mathcal{E}+\mathcal{O},\quad \beta\mathcal{M}=\mathcal{M}\beta,\quad \beta\mathcal{E}=\mathcal{E}\beta,\quad \beta\mathcal{O}=-\mathcal{O}\beta. \tag{17}$$

Here, the operators $\mathcal{M},\mathcal{E}$ are even, and $\mathcal{O}$ is odd.

Foldy-Wouthuysen representation is obtained by means of the unitary transformation

$$\psi_{FW}=U\psi,\quad \mathcal{H}_{FW}=U\mathcal{H}U^{-1}-i\hbar U\partial_t U^{-1}. \tag{18}$$

In arbitrary strong external fields, the following transformation operator can be used [7]:

$$\begin{array}{l} U=\frac{\beta\epsilon+\beta\mathcal{M}-\mathcal{O}}{\sqrt{(\beta\epsilon+\beta\mathcal{M}-\mathcal{O})^2}}\,\beta,\\ U^{-1}=\beta\,\frac{\beta\epsilon+\beta\mathcal{M}-\mathcal{O}}{\sqrt{(\beta\epsilon+\beta\mathcal{M}-\mathcal{O})^2}}. \end{array} \tag{19}$$

Here $\epsilon=\sqrt{\mathcal{M}^2+\mathcal{O}^2}$, and $U^{-1}=U^\dagger$ if $\mathcal{H}=\mathcal{H}^\dagger$. Applying (18), we obtain the explicit transformed Hamiltonian

$$\begin{array}{r} \mathcal{H}'=\beta\epsilon+\mathcal{E}+\frac{1}{2T}\Big(\,[T,[T,(\beta\epsilon+\mathcal{Z})]]\\ +\beta\,[\mathcal{O},[\mathcal{O},\mathcal{M}]]-[\mathcal{O},[\mathcal{O},\mathcal{Z}]]\\ -[(\epsilon+\mathcal{M}),[(\epsilon+\mathcal{M}),\mathcal{Z}]]-[(\epsilon+\mathcal{M}),[\mathcal{M},\mathcal{O}]]\\ -\beta\,\{\mathcal{O},[(\epsilon+\mathcal{M}),\mathcal{Z}]\}+\beta\,\{(\epsilon+\mathcal{M}),[\mathcal{O},\mathcal{Z}]\}\Big)\frac{1}{T}, \end{array} \tag{20}$$

where $\mathcal{Z}=\mathcal{E}-i\hbar\frac{\partial}{\partial t}$ and $T=\sqrt{(\beta\epsilon+\beta\mathcal{M}-\mathcal{O})^2}$. The square and curly brackets denote the commutator $[A,B]=AB-BA$ and the anticommutator $\{A,B\}=AB+BA$, respectively.

The Hamiltonian (20) still contains odd terms proportional to $\hbar$. We can write it as follows:

$$\mathcal{H}'=\beta\epsilon+\mathcal{E}'+\mathcal{O}',\quad \beta\mathcal{E}'=\mathcal{E}'\beta,\quad \beta\mathcal{O}'=-\mathcal{O}'\beta, \tag{21}$$

where $\epsilon=\sqrt{\mathcal{M}^2+\mathcal{O}^2}$. The even and odd parts are determined by

$$\mathcal{E}'=\frac{1}{2}\left(\mathcal{H}'+\beta\mathcal{H}'\beta\right)-\beta\epsilon,\qquad \mathcal{O}'=\frac{1}{2}\left(\mathcal{H}'-\beta\mathcal{H}'\beta\right). \tag{22}$$

Additional unitary transformation removes the odd part, so that the final approximate Hamiltonian reads [7]

$$\mathcal{H}_{FW}=\beta\epsilon+\mathcal{E}'+\frac{1}{4}\beta\left\{\mathcal{O}'^2,\frac{1}{\epsilon}\right\}. \tag{23}$$

For the case under consideration, we have explicitly [12]

$$\mathcal{M} = mc^2V, \tag{24}$$

$$\mathcal{E} = q\Phi+\frac{c}{2}\left(\boldsymbol{K}\cdot\boldsymbol{\pi}+\boldsymbol{\pi}\cdot\boldsymbol{K}\right)+\frac{\hbar c}{4}\,\boldsymbol{\Xi}\cdot\boldsymbol{\Sigma}, \tag{25}$$

$$\mathcal{O} = \frac{c}{2}\left(\pi_b\,\mathcal{F}^b{}_a\alpha^a+\alpha^a\mathcal{F}^b{}_a\pi_b\right)-\frac{\hbar c}{4}\,\Upsilon\gamma_5. \tag{26}$$

Foldy-Wouthuysen Hamiltonian and quantum dynamics

We now limit ourselves to the case when the electromagnetic field is switched off. The computations along the lines described in the previous subsection are straightforward, and after a lengthy algebra we obtain the Foldy-Wouthuysen Hamiltonian in the following form

$$\mathcal{H}_{FW}=\mathcal{H}^{(1)}_{FW}+\mathcal{H}^{(2)}_{FW}. \tag{27}$$

Here the two terms read, respectively [27],

$$\begin{aligned}\mathcal{H}^{(1)}_{FW} &= \beta\epsilon' + \tfrac{\hbar c^2}{16}\Big\{\tfrac{1}{\epsilon'},\big(2\epsilon^{cae}\Pi_e\{p_b,\mathcal{F}^d{}_c\partial_d\mathcal{F}^b{}_a\}\\ &\qquad +\Pi^a\{p_b,\mathcal{F}^b{}_a\Upsilon\}\big)\Big\}\\ &+\tfrac{\hbar mc^4}{4}\epsilon^{cae}\Pi_e\left\{\tfrac{1}{\mathcal{T}},\left\{p_d,\mathcal{F}^d{}_c\mathcal{F}^b{}_a\partial_b V\right\}\right\},\end{aligned} \tag{28}$$

$$\begin{aligned}\mathcal{H}^{(2)}_{FW} &= \tfrac{c}{2}\left(K^a p_a + p_a K^a\right) + \tfrac{\hbar c}{4}\,\Sigma_a\Xi^a\\ &+\tfrac{\hbar c^2}{16}\Bigg\{\tfrac{1}{\mathcal{T}},\Big\{\Sigma_a\{p_e,\mathcal{F}^e{}_b\},\Big\{p_f,\big[\epsilon^{abc}(\tfrac{1}{c}\dot{\mathcal{F}}^f{}_c - \mathcal{F}^d{}_c\partial_d K^f\\ &\quad +K^d\partial_d\mathcal{F}^f{}_c) - \tfrac{1}{2}\mathcal{F}^f{}_d\left(\delta^{db}\Xi^a - \delta^{da}\Xi^b\right)\big]\Big\}\Big\}\Bigg\},\end{aligned} \tag{29}$$

$$\begin{aligned}\epsilon' &= \sqrt{m^2c^4V^2 + \tfrac{c^2}{4}\delta^{ac}\{p_b,\mathcal{F}^b{}_a\}\{p_d,\mathcal{F}^d{}_c\}},\\ \mathcal{T} &= 2\epsilon'^2 + \{\epsilon', mc^2V\}.\end{aligned} \tag{30}$$

Let us derive the equation of motion of spin. As usual, we introduce the polarization operator $\mathbf{\Pi} = \beta\mathbf{\Sigma}$, and the corresponding dynamical equation is obtained from its commutator with the FW Hamiltonian. The computation is straightforward and we find

$$\frac{d\mathbf{\Pi}}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW},\mathbf{\Pi}] = \mathbf{\Omega}_{(1)}\times\mathbf{\Sigma} + \mathbf{\Omega}_{(2)}\times\mathbf{\Pi}. \tag{31}$$

Here the 3-vectors $\mathbf{\Omega}_{(1)}$ and $\mathbf{\Omega}_{(2)}$ are the operators of the angular velocity of spin precessing in the exterior classical gravitational field. Their components read explicitly as follows [27]:

$$\begin{aligned}\Omega^a_{(1)} &= \tfrac{mc^4}{2}\left\{\tfrac{1}{\mathcal{T}},\{p_e,\epsilon^{abc}\mathcal{F}^e{}_b\mathcal{F}^d{}_c\partial_d V\}\right\}\\ &+\tfrac{c^2}{8}\left\{\tfrac{1}{\epsilon'},\{p_e,(2\epsilon^{abc}\mathcal{F}^d{}_b\partial_d\mathcal{F}^e{}_c + \delta^{ab}\mathcal{F}^e{}_b\Upsilon)\}\right\}\end{aligned} \tag{32}$$

and

$$\begin{aligned}\Omega^a_{(2)} &= \tfrac{\hbar c^2}{8}\Bigg\{\tfrac{1}{\mathcal{T}},\Big\{\{p_e,\mathcal{F}^e{}_b\},\Big\{p_f,\big[\epsilon^{abc}(\tfrac{1}{c}\dot{\mathcal{F}}^f{}_c - \mathcal{F}^d{}_c\partial_d K^f\\ &\quad +K^d\partial_d\mathcal{F}^f{}_c) - \tfrac{1}{2}\mathcal{F}^f{}_d\left(\delta^{db}\Xi^a - \delta^{da}\Xi^b\right)\big]\Big\}\Big\}\Bigg\} + \tfrac{c}{2}\,\Xi^a.\end{aligned} \tag{33}$$

One may notice that the two different matrices, $\mathbf{\Sigma}$ and $\mathbf{\Pi}$, appear on the right-hand side of Eq. (31). This is explained by the fact that the vector $\mathbf{\Omega}_{(1)}$ contains odd number of components of the momentum operator, whereas the vector $\mathbf{\Omega}_{(2)}$ contains even number of p_a. Actually, both $\mathbf{\Omega}_{(1)}$ and $\mathbf{\Omega}_{(2)}$ depend only on the combination $\mathcal{F}^b{}_a p_b$. However, the velocity operator is proportional to an additional β factor and is equal to $v_a = \beta c^2\mathcal{F}^b{}_a p_b/\epsilon'$, as we demonstrate below, see Eq. (40). As a result, the operator $\mathbf{\Omega}_{(1)}$ also acquires an additional β factor [11], when it is rewritten in terms of the velocity operator $\boldsymbol{v}$. Note also that only the upper part of β proportional the unit matrix is relevant in the FW representation. Therefore, the appearance of β does not lead to any physical effects at least until antiparticles are considered (which would require special investigations).

We now use the general results above to obtain the corresponding semiclassical expressions by evaluating all anticommutators and neglecting terms of the powers of $\hbar$ higher than 1. Then Eqs. (31)-(33) yield the following explicit semiclassical equations describing the motion of the average spin (as usual, vector product is defined by $\{\boldsymbol{A}\times\boldsymbol{B}\}_a = \epsilon_{abc}A^bB^c$) [27]:

$$\tfrac{d\boldsymbol{s}}{dt} = \mathbf{\Omega}\times\boldsymbol{s} = (\mathbf{\Omega}_{(1)} + \mathbf{\Omega}_{(2)})\times\boldsymbol{s}, \tag{34}$$

$$\Omega^a_{(1)} = \frac{c^2}{\epsilon'}\mathcal{F}^d{}_c p_d\Big(\frac{1}{2}\Upsilon\delta^{ac} - \epsilon^{aef}V\mathcal{C}_{ef}{}^c + \frac{\epsilon'}{\epsilon'+mc^2V}\epsilon^{abc}W^e{}_{\widehat{b}}\,\partial_e V\Big), \tag{35}$$

$$\Omega^a_{(2)} = \frac{c}{2}\,\Xi^a - \frac{c^3}{\epsilon'(\epsilon'+mc^2V)}\,\epsilon^{abc}Q_{(bd)}\delta^{dn}\mathcal{F}^k{}_n p_k\mathcal{F}^l{}_c p_l. \tag{36}$$

Here, in the semiclassical limit,

$$\epsilon' = \sqrt{m^2c^4V^2 + c^2\delta^{cd}\mathcal{F}^a{}_c\,\mathcal{F}^b{}_d\,p_a\,p_b}\,. \tag{37}$$

We can substitute the results obtained into the FW Hamiltonian (27) and recast the latter in a compact and transparent form in terms of the precession angular velocities $\mathbf{\Omega}_{(1)}, \mathbf{\Omega}_{(2)}$:

$$\mathcal{H}_{FW} = \beta\epsilon' + \frac{c}{2}\,(\boldsymbol{K}\cdot\boldsymbol{p} + \boldsymbol{p}\cdot\boldsymbol{K}) + \frac{\hbar}{2}\big(\mathbf{\Pi}\cdot\mathbf{\Omega}_{(1)} + \mathbf{\Sigma}\cdot\mathbf{\Omega}_{(2)}\big). \tag{38}$$

We can use Eq. (38) to derive the velocity operator in the semiclassical approximation:

$$\frac{dx^a}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW}, x^a] = \beta\,\frac{\partial\epsilon'}{\partial p_a} + cK^a = \beta\,\frac{c^2}{\epsilon'}\,\mathcal{F}^a{}_b\delta^{bc}\mathcal{F}^d{}_c p_d + cK^a. \tag{39}$$

Comparing this with the relation between the holonomic and anholonomic components of the velocity, Eq. (6), we find the velocity operator in the Schwinger frame (2):

$$\beta\,\frac{c^2}{\epsilon'}\,\mathcal{F}^b{}_a p_b = v_a. \tag{40}$$

This immediately yields $\delta^{cd}\mathcal{F}^a{}_c\,\mathcal{F}^b{}_d p_a p_b = (\epsilon')^2 v^2/c^2$. Using this in (37), we have $(\epsilon')^2 = m^2c^4V^2 + (\epsilon')^2v^2/c^2$, and thus we find

$$\epsilon' = \gamma\, mc^2\, V. \tag{41}$$

Equations (40) and (41) are crucial for establishing the full agreement of the quantum and classical dynamics of spin. In particular, using (40) and (41), we find

$$\frac{\epsilon'}{\epsilon'+mc^2V} = \frac{\gamma}{1+\gamma}, \qquad \frac{c^3}{\epsilon'(\epsilon'+mc^2V)}\,\mathcal{F}^b{}_a p_b\mathcal{F}^d{}_c p_d = \frac{\gamma}{1+\gamma}\,\frac{v_a v_c}{c}. \tag{42}$$

Quantum-mechanical equations of particle dynamics

We now turn to the quantum-mechanical analysis of a particle in the gravitational field. The dynamics of spin is described in an anholonomic frame. For consistency, we will use an anholonomic frame description for the particle dynamics, too. The Schwinger gauge with $e^0_{\widehat{a}} = 0$ simplifies the equation for the force operator which is given by [27]

$$F_{\widehat{a}} = \frac{dp_{\widehat{a}}}{dt} = \frac{1}{2}\frac{d}{dt}\left\{e^b_{\widehat{a}}, p_b\right\} = \frac{1}{2}\left\{\frac{dW^b{}_{\widehat{a}}}{dt}, p_b\right\} + \frac{1}{2}\left\{W^b{}_{\widehat{a}}, \frac{dp_b}{dt}\right\} = \frac{1}{2}\left\{\dot{W}^b{}_{\widehat{a}}, p_b\right\} + \frac{i}{2\hbar}\left\{[\mathcal{H}_{FW}, W^b{}_{\widehat{a}}], p_b\right\} - \frac{1}{2}\left\{W^b{}_{\widehat{a}}, \partial_b\mathcal{H}_{FW}\right\}. \tag{43}$$

Here as before the partial derivative with respect to the coordinate time is denoted by the dot, in particular, $\dot{W}^b{}_{\widehat{a}} := \partial_t W^b{}_{\widehat{a}}$.

The explicit expression for the force operator reads [27]

$$F_{\widehat{a}} = \tfrac{1}{2}\left\{\dot{W}^{b}{}_{\widehat{a}}, p_b\right\} + \tfrac{1}{4}\left\{p_b, \left\{\tfrac{\partial \mathcal{H}_{FW}}{\partial p_c}, \partial_c W^{b}{}_{\widehat{a}}\right\}\right\} - \tfrac{1}{2}\left\{W^{b}{}_{\widehat{a}}, \partial_b \mathcal{H}_{FW}\right\}, \tag{44}$$

$$\tfrac{\partial \mathcal{H}_{FW}}{\partial p_c} = \beta \tfrac{c^2}{4}\delta^{ad}\left\{\tfrac{1}{\epsilon'}, \left\{p_b, \mathcal{F}^{b}{}_{a}\mathcal{F}^{c}{}_{d}\right\}\right\} + cK^c + \tfrac{\hbar}{2}\mathfrak{T}^c, \tag{45}$$

where we introduced the following compact notation

$$\mathfrak{T}^c = \frac{\partial \mathcal{U}}{\partial p_c}, \qquad \mathcal{U} := \boldsymbol{\Pi}\cdot\boldsymbol{\Omega}_{(1)} + \boldsymbol{\Sigma}\cdot\boldsymbol{\Omega}_{(2)}. \tag{46}$$

Corrections due to the noncommutativity of operators are of order of $\hbar^2$ and can be neglected in (44). Let us split the total force operator into the terms of the zeroth and first orders in the Planck constant:

$$F_{\widehat{a}} = F^{(0)}_{\widehat{a}} + F^{(1)}_{\widehat{a}}. \tag{47}$$

The zeroth order terms read as follows

$$F^{(0)}_{\widehat{a}} = \tfrac{1}{2}\left\{\dot{W}^{b}{}_{\widehat{a}}, p_b\right\} - \tfrac{1}{2}\Bigg\{W^{b}{}_{\widehat{a}}, \partial_b\Big[\beta\epsilon' + \tfrac{c}{2}\big(K^a p_a + p_a K^a\big)\Big]\Bigg\} + \tfrac{1}{4}\Bigg\{p_b, \Big\{\Big(\beta\tfrac{c^2}{4}\delta^{ad}\big\{\tfrac{1}{\epsilon'}, \{p_b, \mathcal{F}^{b}{}_{a}\mathcal{F}^{c}{}_{d}\}\big\} + cK^c\Big), \partial_c W^{b}{}_{\widehat{a}}\Big\}\Bigg\}. \tag{48}$$

These terms describe the influence of the gravitational field on the particle without taking into account its internal structure. The first term in Eq. (48) is important for the motion of the particle in nonstationary spacetimes, for example, in cosmological context. The next term describes the Newtonian force, the related relativistic corrections, and the Coriolis-like force proportional to $\boldsymbol{K}$. The last term also contributes to the relativistic corrections to the force acting in static spacetimes that arise in addition to the velocity-independent Newtonian force.

All the terms of the first order in the Planck constant are proportional to the spin operators and therefore they collectively represent the quantum-mechanical counterpart of the Mathisson force (which is an analogue of the Stern-Gerlach force in electrodynamics). This force is given by, recall the notation (46),

$$F^{(1)}_{\widehat{a}} = \frac{\hbar}{8}\left\{p_b, \left\{\mathfrak{T}^c, \partial_c W^{b}{}_{\widehat{a}}\right\}\right\} - \frac{\hbar}{4}\left\{W^{b}{}_{\widehat{a}}, \partial_b \mathcal{U}\right\}. \tag{49}$$

We will demonstrate the agreement between the quantum-mechanical and the classical equations of particle dynamics in the next section.

Equation (48) and (49) perfectly reproduce all previously obtained quantum-mechanical results [8,9,11,12]. In order to illustrate this, let us find the force on the spinning particle in the metric [19] of an arbitrarily moving noninertial (accelerated and rotating) observer:

$$V = 1 + \frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2}, \quad W^{\widehat{a}}{}_{b} = \delta^{a}_{b}, \quad K^a = -\frac{1}{c}\left(\boldsymbol{\omega}\times\boldsymbol{r}\right)^a. \tag{50}$$

The FW Hamiltonian for this metric derived in Ref. [12] reads:

$$\begin{aligned}\mathcal{H}_{FW} &= \frac{\beta}{2}\left\{\left(1 + \frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2}\right), \sqrt{m^2c^4 + c^2\boldsymbol{p}^2}\right\} - \boldsymbol{\omega}\cdot(\boldsymbol{r}\times\boldsymbol{p}) \\ &\quad + \frac{\hbar}{2}\boldsymbol{\Pi}\cdot\frac{\boldsymbol{a}\times\boldsymbol{p}}{mc^2(\gamma+1)} - \frac{\hbar}{2}\boldsymbol{\Sigma}\cdot\boldsymbol{\omega},\end{aligned} \tag{51}$$

where the object that has the meaning of the Lorentz factor is defined by

$$\gamma = \frac{\sqrt{m^2c^4 + c^2\boldsymbol{p}^2}}{mc^2}. \tag{52}$$

Using the FW Hamiltonian (51) in (44) and (45) yields the explicit force

$$\begin{aligned} F_{\widehat{a}} &= -\frac{\beta}{c^2}\, a_a \sqrt{m^2c^4 + c^2\boldsymbol{p}^2} - (\boldsymbol{\omega}\times\boldsymbol{p})_a \\ &= \beta m\gamma \left(-\boldsymbol{a} + \boldsymbol{v}\times\boldsymbol{\omega}\right)_a . \end{aligned} \tag{53}$$

Here we used Eq. (52) and the relation between the operators of momentum and velocity $p_{\widehat{a}} = e^b_{\widehat{a}} p_b = \beta\gamma m v_a$ which is recovered from (40). One can straightforwardly verify that the usual structure of the inertial forces (in particular, the Coriolis and centrifugal pieces) is encoded in the force (53), see the corresponding computation of the coordinate acceleration operator in [12].

For the metric (50), the spacetime curvature vanishes. As a result, the curvature- and spin-dependent Mathisson force is zero. In the general case, the Mathisson force is nontrivial and violates the equivalence principle (see Ref. [28]). In a separate publication, we will analyse the possible generalization of the equivalence principle for spinning particles, making use of the force framework developed here. As a preliminary step, we refer to [12] where we evaluated the quantum force for the weak gravitational field and recovered the linearized Mathisson force, thus confirming the earlier results [29, 30].

Any theory based on the Dirac equation can reproduce only a certain reduced form of the equation of spin motion. The formal reason is the absence in the Lagrangian and the Hamiltonian of the terms bilinear in the spin matrices because their product can always be simplified: $\Sigma^a\Sigma^b = \delta^{ab} + i\epsilon^{abc}\Sigma^c$. As a result, the equation of spin motion of a Dirac particle cannot contain such terms. In quantum mechanics of particles with higher spins ($s > 1/2$) as well as in the classical gravity, the terms bilinear in spin cannot be reduced and the general MP equations [31, 32] should be used.

Classical spinning particles

Mathisson-Papapetrou approach

The motion of classical spinning particles in the gravitational field can be consistently described by the generally covariant MP theory [31, 32], for the recent discussion see also [33–36]. In this framework, a test particle is characterized by the 4-velocity U^α and the tensor of spin $S^{\alpha\beta} = -S^{\beta\alpha}$. The total 4-momentum is not collinear with the velocity, in general. In Ref. [37], a different noncovariant approach was developed, in which the main dynamical variable is the 3-dimensional spin defined in the rest frame of a particle. In our previous work [8–12] we have used the MP theory and demonstrated its consistency with the noncovariant approach.

The analysis of the general MP equations is a difficult task [35] and the exact solutions are not available even for the simple spacetime geometries. The knowledge of the symmetries of the gravitational field, i.e., of the corresponding Killing vectors, significantly helps in the integration of the equations of motion, as can be demonstrated [38] for the de Sitter spacetime, in particular. However, in the absence of the symmetries, various approximation schemes were developed to find solutions of MP equations of motion. Following [33], we neglect the second order spin effects, so that the MP system reduces to

$$\frac{DU^\alpha}{d\tau} = f^\alpha_{\rm m} = -\frac{1}{2m}\, S^{\mu\nu}U^\beta R_{\mu\nu\beta}{}^\alpha, \tag{54}$$

$$\frac{DS^{\alpha\beta}}{d\tau} = \frac{U^\alpha U_\gamma}{c^2}\frac{DS^{\gamma\beta}}{d\tau} - \frac{U^\beta U_\gamma}{c^2}\frac{DS^{\gamma\alpha}}{d\tau}. \tag{55}$$

On the right-hand side of (54) we have the Mathisson force $f^\alpha_{\rm m}$ that depends on the Riemann curvature tensor $R_{\mu\nu\beta}{}^\alpha$ of spacetime. The tensor of spin satisfies the Frenkel condition $U_\alpha S^{\alpha\beta} = 0$ and gives rise to the vector of spin

$$S_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma}S^{\beta\gamma}. \tag{56}$$

Here we use the totally antisymmetric tensor

$$\epsilon_{\alpha\beta\gamma} = \frac{1}{c}\eta_{\alpha\beta\gamma\delta}U^\delta, \tag{57}$$

constructed from the Levi-Civita tensor $\eta_{\alpha\beta\gamma\delta}$. The relation (56) can be inverted

$$S^{\alpha\beta} = -\epsilon^{\alpha\beta\gamma} S_\gamma \tag{58}$$

with the help of the identity

$$\epsilon^{\alpha\beta\gamma}\epsilon_{\mu\nu\gamma} = P^\alpha_\nu P^\beta_\mu - P^\alpha_\mu P^\beta_\nu, \tag{59}$$

where $P^\alpha_\mu = \delta^\alpha_\mu - \frac{1}{c^2} U^\alpha U_\mu$ is the projector on the rest frame of the particle.

Using the definition (56), we rewrite the equation (55) in an alternative form

$$\frac{DS_\alpha}{d\tau} = \frac{U_\alpha U^\beta}{c^2}\frac{DS_\beta}{d\tau} = -\frac{1}{c^2} U_\alpha f^\beta_{\rm m} S_\beta. \tag{60}$$

With the help of the projectors and antisymmetric tensor, one can decompose the curvature tensor into the three irreducible pieces

$$\mathbb{E}_{\alpha\beta} = \frac{U^\mu U^\nu}{c^2} R_{\alpha\mu\beta\nu}, \tag{61}$$

$$\mathbb{M}^{\alpha\beta} = \frac{1}{4}\epsilon^{\alpha\mu\nu}\epsilon^{\beta\rho\sigma} R_{\mu\nu\rho\sigma}, \tag{62}$$

$$\mathbb{B}_\alpha{}^\beta = \frac{U_\gamma}{2c}\epsilon_{\alpha\mu\nu} R^{\beta\gamma\mu\nu}. \tag{63}$$

By construction, these tensors satisfy the orthogonality conditions
$\mathbb{E}_{\alpha\beta}U^\beta = 0$, $\mathbb{M}^{\alpha\beta}U_\beta = 0$, $\mathbb{B}_\alpha{}^\beta U_\beta = 0$, $\mathbb{B}_\alpha{}^\beta U^\alpha = 0$. Taking into account the obvious symmetry $\mathbb{E}_{\alpha\beta} = \mathbb{E}_{\beta\alpha}$ and $\mathbb{M}^{\alpha\beta} = \mathbb{M}^{\beta\alpha}$, we have $6+6+8 = 20$ independent components for these objects. The curvature decomposition reads explicitly

$$\begin{array}{l} R^{\alpha\beta\mu\nu} = \frac{1}{c^2}\big(U^\alpha U^\mu \mathbb{E}^{\beta\nu} - U^\beta U^\mu \mathbb{E}^{\alpha\nu} - U^\alpha U^\nu \mathbb{E}^{\beta\mu} \\ \quad + U^\beta U^\nu \mathbb{E}^{\alpha\mu}\big) + \epsilon^{\alpha\beta\gamma}\epsilon^{\mu\nu\lambda}\mathbb{M}_{\gamma\lambda} + \frac{1}{c}\big[\epsilon^{\alpha\beta\gamma}(U^\mu \mathbb{B}_\gamma{}^\nu \\ \qquad - U^\nu \mathbb{B}_\gamma{}^\mu) + \epsilon^{\mu\nu\gamma}\left(U^\alpha \mathbb{B}_\gamma{}^\beta - U^\beta \mathbb{B}_\gamma{}^\alpha\right)\big]. \end{array} \tag{64}$$

As a result, we rewrite the Mathisson force as [27]

$$f^\alpha_{\rm m} = \frac{c}{2m}\mathbb{B}_\beta{}^\alpha S^\beta. \tag{65}$$

The physical spin is defined in the rest frame of a particle where the 4-velocity reduces to $u^\alpha = (1, \mathbf{0}) = \delta^\alpha_0$. The local reference frame and the rest frame are related by the Lorentz transformation such that $U^\alpha = \Lambda^\alpha{}_\beta u^\beta$. Recalling $U^\alpha = (\gamma, \gamma v^a)$, the Lorentz matrix reads explicitly

$$\Lambda^\alpha{}_\beta = \left(\begin{array}{c|c} \gamma & \gamma v_b/c^2 \\ \hline \gamma v^a & \delta^a_b + (\gamma-1)v^a v_b/v^2 \end{array}\right), \tag{66}$$

with the Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$, where $v^2 = \delta_{ab} v^a v^b$.

The physical spin is then $s^\alpha = (\Lambda^{-1})^\alpha{}_\beta S^\beta$, hence $s^\alpha = (0, \boldsymbol{s})$. We rewrite Eq. (60) as $\frac{dS^\alpha}{d\tau} = \Phi^\alpha{}_\beta S^\beta$, with $\Phi^\alpha{}_\beta = -U^i \Gamma_{i\beta}{}^\alpha + \frac{1}{c^2}(f^\alpha_{\rm m} U_\beta - f_{\beta{\rm m}} U^\alpha)$. From this we find the equation of motion of the physical spin:

$$\frac{ds^\alpha}{d\tau} = \Omega^\alpha{}_\beta s^\beta, \tag{67}$$

$$\Omega^\alpha{}_\beta = (\Lambda^{-1})^\alpha{}_\gamma \Phi^\gamma{}_\delta \Lambda^\delta{}_\beta - (\Lambda^{-1})^\alpha{}_\gamma \frac{d}{d\tau}\Lambda^\gamma{}_\beta. \tag{68}$$

Noticing that with respect to the coordinate basis the 4-velocity is $U^i = \gamma e^i_{\hat{0}} + \gamma v^a e^i_{\hat{a}}$, we recast the MP system (54) and (67) into the 3-vector form [27]

$$\frac{d\gamma}{d\tau} = \frac{\gamma}{c^2}\,\boldsymbol{v}\cdot\widehat{\boldsymbol{\mathcal{E}}}, \tag{69}$$

$$\frac{d(\gamma \boldsymbol{v})}{d\tau} = \gamma\left(\widehat{\boldsymbol{\mathcal{E}}} + \boldsymbol{v}\times\boldsymbol{\mathcal{B}}\right), \tag{70}$$

$$\frac{d\boldsymbol{s}}{d\tau} = \boldsymbol{\Omega}\times\boldsymbol{s}. \tag{71}$$

Here using Eqs. (1) and (65), we introduced the objects that can be called the generalized gravitoelectric and gravitomagnetic fields [27]:

$$\mathcal{E}^a = \frac{\gamma}{V}\delta^{ac}\left(c\mathcal{Q}_{(\widehat{c}\widehat{b})}v^b - c^2\, W^b{}_{\widehat{c}}\,\partial_b V\right), \tag{72}$$

$$\mathcal{B}^a = \frac{\gamma}{V}\left(-\frac{c}{2}\,\Xi^a - \frac{1}{2}\Upsilon\, v^a + \epsilon^{abc}V\mathcal{C}_{bc}{}^d v_d\right), \tag{73}$$

$$\widehat{\boldsymbol{\mathcal{E}}}^a = \boldsymbol{\mathcal{E}}^a + \frac{c}{2m\gamma}I\!B_b{}^a\left(s^b - \frac{\gamma}{\gamma+1}\frac{v^b v_c}{c^2}s^c\right). \tag{74}$$

The components of the angular velocity of the spin precession, $-\epsilon^{abc}\Omega_{bc}/2$, are obtained from Eq. (68):

$$\boldsymbol{\Omega} = -\boldsymbol{\mathcal{B}} + \frac{\gamma}{\gamma+1}\,\frac{\boldsymbol{v}\times\boldsymbol{\mathcal{E}}}{c^2}. \tag{75}$$

Alternatively, we can explicitly write the precession velocity components with the help of (9) and (10) as [11,37]

$$\Omega_{\widehat{a}} = \epsilon_{abc}\,U^i\left(\frac{1}{2}\Gamma_i{}^{\widehat{c}\widehat{b}} + \frac{\gamma}{\gamma+1}\,\Gamma_{i\widehat{0}}{}^{\widehat{b}}v^{\widehat{c}}/c^2\right). \tag{76}$$

Finally, substituting (72) and (73) into (75), we obtain the *exact classical formula* for the angular velocity of the spin precession in an arbitrary gravitational field [27]:

$$\Omega^{\widehat{a}} = \frac{\gamma}{V}\left(\frac{1}{2}\Upsilon\, v^{\widehat{a}} - \epsilon^{abc}V\mathcal{C}_{\widehat{b}\widehat{c}}{}^d v_{\widehat{d}} + \frac{\gamma}{\gamma+1}\epsilon^{abc}W^d{}_{\widehat{b}}\,\partial_d V v_{\widehat{c}}\right.$$
$$\left. + \frac{c}{2}\,\Xi^{\widehat{a}} - \frac{\gamma}{\gamma+1}\epsilon^{abc}\mathcal{Q}_{(\widehat{b}\widehat{d})}\frac{v^{\widehat{d}}v_{\widehat{c}}}{c}\right). \tag{77}$$

The terms in the first line are linear in the 4-velocity of the particle, whereas the terms in the second line contain the even number of the velocity factors.

As compared to the precession of the quantum spin described by $\boldsymbol{\Omega}^{(1)}$ and $\boldsymbol{\Omega}^{(2)}$ using the coordinate time, the classical spin precession velocity $\Omega^{\widehat{a}}$ contains an extra factor

$$\frac{dt}{d\tau} = U^0 = \frac{\gamma}{V}, \tag{78}$$

since the classical dynamics is parameterized using the proper time.

It is worthwhile to notice that the equations of motion of a particle (69) and (70) have a remarkably simple form of the motion of a relativistic charged particle under the action of the Lorentz force. It is interesting to mention a certain asymmetry: the Mathisson force (65), that depends on the spin and the curvature of spacetime, contributes only to the gravitoelectric field (74) but not to the gravitomagnetic one. Using Eq. (69), we can recast Eq. (70) into the dynamical equation [27]

$$\frac{d\boldsymbol{v}}{d\tau} = \widehat{\boldsymbol{\mathcal{E}}} - \frac{\boldsymbol{v}(\boldsymbol{v}\cdot\widehat{\boldsymbol{\mathcal{E}}})}{c^2} + \boldsymbol{v}\times\boldsymbol{\mathcal{B}}. \tag{79}$$

Let us consider the motion of the classical particle in the metric of a noninertial observer (50). Since $\widehat{\boldsymbol{\mathcal{E}}} = \boldsymbol{\mathcal{E}} = -\frac{\gamma}{V}\,\boldsymbol{a}$ and $\boldsymbol{\mathcal{B}} = \frac{\gamma}{V}\,\boldsymbol{\omega}$,

$$\frac{d(\gamma\boldsymbol{v})}{dt} = \gamma\left(-\boldsymbol{a} + \boldsymbol{v}\times\boldsymbol{\omega}\right), \tag{80}$$

where we changed from the proper time parametrization to the coordinate time using (78). As we see, the classical (80) and the quantum (53) forces are the same.

Finally, making use of (40) and (41), we conclude that the classical equation of the spin motion (75) agrees with the quantum equation (31) and with the semiclassical one (34). Thus, the classical and the quantum theories of the spin motion in gravity are in complete agreement. This is now verified for the *arbitrary gravitational field* configurations. We thus confirm and extend our previous results obtained for the weak fields [11] and for special strong field configurations [12].

Hamiltonian approach

It is instructive to compare the classical and quantum Hamiltonians of a spinning particle. In order to do this, one can start from the classical Hamiltonian of a spinless relativistic point particle (with an electric charge q, in general). With the 3-velocity $v^a = dx^a/dt = U^a/U^0$, we can write the Lagrangian in the form

$$\mathcal{L} = -mc\left(g_{00} + 2g_{0a}v^a + g_{ab}v^a v^b\right)^{1/2} - qA_0 - qA_b v^b. \tag{81}$$

The canonical momentum is

$$p_a = \frac{\partial \mathcal{L}}{\partial v^a} = -\frac{mc(g_{0a} + g_{ab}v^b)}{(g_{00} + 2g_{0a}v^a + g_{ab}v^a v^b)^{1/2}} - qA_a. \tag{82}$$

Inverting, we find velocity in terms of momentum $\pi_a = p_a + qA_a$:

$$v^a = \frac{g^{0a}}{g^{00}} - \frac{\tilde{g}^{ab}\pi_b}{[g^{00}(m^2c^2 - \tilde{g}^{ab}\pi_a\pi_b)]^{1/2}}, \quad \tilde{g}^{ab} = g^{ab} - \frac{g^{0a}g^{0b}}{g^{00}}. \tag{83}$$

As a result, the classical Hamiltonian reads (see Ref. [39])

$$\mathcal{H}_{class} = p_a v^a - \mathcal{L} = \sqrt{\frac{m^2c^2 - \tilde{g}^{ab}\pi_a\pi_b}{g^{00}}} + \frac{g^{0a}\pi_a}{g^{00}} + qA_0. \tag{84}$$

For the contravariant components of the general metric (1) we have $g^{ij} = e^i_\alpha e^j_\beta g^{\alpha\beta} = \frac{1}{c^2}e^i_{\hat{0}}e^j_{\hat{0}} - e^i_{\hat{c}}e^j_{\hat{d}}\delta^{cd}$. Thus explicitly, using Eq. (3):

$$\begin{array}{l} g^{00} = \frac{1}{c^2V^2}, \quad g^{0a} = \frac{K^a}{cV^2}, \\ g^{ab} = \frac{1}{V^2}\left(K^aK^b - \mathcal{F}^a{}_c\mathcal{F}^b{}_d\delta^{cd}\right). \end{array} \tag{85}$$

As a result, the classical Hamiltonian (84) takes the form

$$\mathcal{H}_{class} = \sqrt{m^2c^4V^2 + c^2\delta^{cd}\mathcal{F}^a{}_c\mathcal{F}^b{}_d\,\pi_a\,\pi_b} + c\boldsymbol{K}\cdot\boldsymbol{\pi} + q\Phi. \tag{86}$$

Now, let us discuss a generalization of the Hamiltonian theory with spin included. In order to take into account the spin correctly, in a Cosserat type approach a material frame (of four linearly independent vectors) is attached to a particle, thus modelling its internal rotational degrees of freedom. We denote it h^i_α.

Such a material frame does not coincide with the spacetime frame, $h^i_\alpha \neq e^i_\alpha$. In particular, the zeroth leg is given by particle's 4-velocity

$$h^i_{\hat{0}} = U^i. \tag{87}$$

Any two orthonormal frames are related by a Lorentz transformation, $h^i_\alpha = e^i_\beta\Lambda^\beta{}_\alpha$. The condition (87) means that the Lorentz matrix $\Lambda^\beta{}_\alpha$ brings one to a local reference frame $U^\alpha = \Lambda^\alpha{}_\beta u^\beta$ in which the particle is at rest, i.e., $u^\alpha = \delta^\alpha_{\hat{0}}$. This is straightforwardly demonstrated: $U^i = e^i_\alpha U^\alpha = e^i_\alpha\Lambda^\alpha{}_\beta u^\beta = h^i_\alpha u^\alpha = h^i_{\hat{0}}$. The corresponding Lorentz transformation is explicitly given by Eq. (66).

The standard way to take the dynamics of spin into account [40–42] is to amend the classical Hamiltonian by the term $\frac{1}{2}S^{ij}\Omega_{ij}$ with

$$\Omega^i{}_j := h^i_\alpha\frac{D}{d\tau}h^\alpha_j = h^i_\alpha U^k\nabla_k h^\alpha_j = h^i_\alpha U^k\left(\partial_k h^\alpha_j - \Gamma_{kj}{}^l h^\alpha_l\right). \tag{88}$$

Rewriting everything in terms of the objects in particle's rest frame, $S^{\alpha\beta} = h^\alpha_i h^\beta_j S^{ij}$ and $\Omega^\alpha{}_\beta = h^\alpha_i h^j_\beta\Omega^i{}_j$, we find

$$\frac{1}{2}S^{ij}\Omega_{ij} = \frac{1}{2}S^{\alpha\beta}\Omega_{\alpha\beta} = \boldsymbol{s}\cdot\boldsymbol{\Omega}. \tag{89}$$

Here we recover the precession velocity vector (76).

The resulting complete Hamiltonian has the structure that was proposed in the framework of the general discussion in the Ref. [37]:

$$\begin{aligned}\mathcal{H}_{class} = &\sqrt{m^2c^4V^2 + c^2\delta^{cd}\mathcal{F}^a{}_c\,\mathcal{F}^b{}_d\,\pi_a\,\pi_b} \\ &+c\boldsymbol{K}\cdot\boldsymbol{\pi} + q\Phi + \boldsymbol{s}\cdot\boldsymbol{\Omega}.\end{aligned} \tag{90}$$

In the general case, $\boldsymbol{\Omega}$ should include both electromagnetic and gravitational contributions.

The obvious similarity of quantum (38) and classical (90) Hamiltonians is another demonstration of complete agreement of the quantum-mechanical and classical equations of motion discussed in the previous subsection. The consistency between the classical Hamiltonian dynamics and the quantum-mechanical equations of particle dynamics derived in Sec. is also confirmed by the computation of the force. Switching off the electromagnetic field, we find the classical equation for the force

$$F^{class}_{\hat{a}} = p_b\dot{W}^b{}_{\hat{a}} + p_b\,\tfrac{\partial\mathcal{H}_{class}}{\partial p_c}\,\partial_c W^b{}_{\hat{a}} - W^b{}_{\hat{a}}\partial_b\mathcal{H}_{class}. \tag{91}$$

Rewriting the spin-dependent part in Eq.(44) in terms of the spin operator, $\boldsymbol{s} = \hbar\boldsymbol{\Sigma}/2$, shows its consistency with Eq. (91).

Conclusions

This paper continues the study of the motion of Dirac fermions in a curved spacetimes. We use the new mathematical method allowing a derivation of a general relativistic FW Hamiltonian [7]. We consider the case of an arbitrary spacetime metric and generalize our results obtained for the weak fields and for the special static and stationary field configurations [8–12, 27]. The convenient parametrization in terms of the functions $V(t,x^c)$, $K^a(t,x^c)$, and $W^{\widehat{a}}{}_b(t,x^c)$ provides a unified description of all possible inertial and gravitational fields. We also include the classical electromagnetic field for completeness. In this general framework, we derive the Hermitian Dirac Hamiltonian (15). Starting with this master equation, we apply the Foldy-Wouthuysen transformation [7] and construct the Hamiltonian (27) in the FW representation for an *arbitrary spacetime geometry*. In this paper, we have confined ourselves to the purely Riemannian case of the Einstein's general relativity. A possible generalization to the non-Riemannian geometries may be analysed elsewhere. Making use of the FW Hamiltonian, we derive the operator equations of motion. In particular, we study the quantum-mechanical spin precession (31) and its semiclassical limit (34). One can apply these general results to compare the dynamics of a spinning particle in the inertial and gravitational fields, thus revisiting the validity of the equivalence principle [43]. We derive the force operator and analyse the quantum dynamics of the particle under its action in Sec. . In the second part of the paper, we consider the motion of the classical particle with spin. In the framework of the Mathisson-Papapetrou theory, we obtained the dynamical equations (69), (70), and (79) which have a remarkably simple form of the motion of a relativistic particle under the action of the Lorentz force, with the Mathisson force included into the generalized gravitoelectric field (74). We also derived the equation (77) for the angular velocity of spin precession in the general gravitational field. It is satisfactory to see that our results further confirm the earlier conclusions [11, 12] and demonstrate that the classical spin dynamics is fully consistent with the semiclassical quantum dynamics of the Dirac fermion. Finally, the complete consistency of the quantum-mechanical and classical descriptions of spinning particles is also established using the Hamiltonian approach in Sec. .

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Space with spinor structure and analytical properties of the solutions of Klein-Fock equation in cylindric parabolic coordinates

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Abstract: *Possible quantum mechanical corollaries of changing the vectorial geometrical model of the physical space, extending it twice, in order to describe its spinor structure (in other terminology and emphasis it is known as the Hopf's bundle) are investigated. The extending procedure is realized in cylindrical parabolic coordinates:* $G(t,u,v,z) \Longrightarrow \tilde{G}(t,u,v,z)$. *It is done through expansion twice as much of the domain* G *so that instead of the half plane* $(u,v>0)$ *now the entire plane* (u,v) *should be used accompanied with new identification rules over the boundary points. Solutions of the Klein-Fock and Schrödinger equations* $\Psi_{\epsilon,p,\,a} = e^{i\epsilon t}e^{ipz}U_a(u)V_a(v)$ *are constructed in terms of parabolic cylinder functions,* a *is a separating constant. Given quantum numbers* ϵ, p, a *four types of solutions are possible:* $\Psi_{++},\Psi_{--};\ \Psi_{+-},\Psi_{-+}$. *The first two* Ψ_{++},Ψ_{--} *provide us with single-valued functions of the vectorial space points, whereas last two* Ψ_{+-},Ψ_{-+} *have discontinuities in the frame of vectorial space and therefore they must be rejected in this model. All four types of functions are continuous ones being regarded in the spinor space. It is shown that solutions* $\Psi_{++},\Psi_{--},\ \Psi_{+-},\Psi_{-+}$ *all are the eigen-functions of two discrete spinor operators* $\hat{\delta}$ *and* $\hat{\pi}$: $\hat{\delta}\ (u,v)=(-u,-v)$, $\hat{\pi}(u,v)=(u,-v)$, $\hat{\delta}\ (x,y)=(x,y)$, $\hat{\pi}(x,y)=(x,-y)$. *Two other classifications of the wave functions over discrete quantum numbers are given. It is established that all solutions* $\Psi_{++},\Psi_{--},\ \Psi_{+-},\Psi_{-+}$ *are orthogonal to each other provided that integration is done over extended domain parameterizing the spinor space. Simple selection rules for matric elements of the vector and spinor coordinates,* (x,y) *and* (u,v), *respectively, are derived. Selection rules for* (u,v) *are substantially different in vector and spinor spaces. In the supplement some relationships describing primary geometric objects, spatial spinor* ξ *and* η, *as functions of cylindrical parabolic coordinates, are given.*

Key words: geometry, spinor space, quantum mechaincs, wave function
PASC: 02.30.Gp, 02.40.Ky, 03.65Ge, 04.62.+v
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Introduction

In the literature, there exist [1-31] three terminological different approaches though close in their intrinsic essence. There are a space-time spinor structure (see the book by by Penrose and Rindler [29] as a modern embodiment of the old idea [1-4] to use spinor groups instead of the orthogonal ones), the Hopf bundle [5] and the Kustaanheimo-Stifel bundle [6,7].

The differences between three mentioned formalisms consist mainly in conceptual accents (see for more detail [32]). In the Hopf's technique it is suggested to use in all parts only complex spinors ξ and conjugated ξ^* instead of real-valued vector (tensor) quantities. In the Kustaanheimo-Stifel approach we are to use four real-valued coordinates, form which Cartesian coordinates (x,y,z) can be formed up by means of definite bilinear functions. These four variables by Kustaanheimo-Stifel are real and imaginary parts of two spinor components. The known spinor invariant $(\xi'\xi^{'*}+\xi^2\xi^{'2})$ becomes the sum of four squared real quantities, so that we can associate spinor technique with geometry of the Riemann space S_3 of constant positive curvature.

In essence, the Kustaanheimo-Stifel's approach is other elaboration of the same Hopf's technique based on complex spinors ξ and ξ^*, in terms of four real-valued variables. In so doing, we are able to hide in the formalism the presence of the non-analytical operation of complex conjugation. Spinor space structure, formalism developed in the present work, also exploits possibilities given by spinors to construct 3-vectors, however the emphasis is taken to doubling the set of spatial points so that we get an extended space model that is called a space with spinor structure [32-38]. In such an extended space, in place of 2π-rotation, only 4π-rotation transfers the space into itself.

The procedure itself of doubling the manyfold can be realized easier when for parameterizing the space some curvilinear coordinate system is used instead of the Cartesian coordinates. In such context, spherical and parabolic coordinates were considered in [37]. In the present paper, the use of cylindrical parabolic coordinates is studied as applied for description of spinor space structure. Now we study analytical properties of Schrödinger and Klein-Fock the wave solutions depending on vector and spinor space models. It is demonstrated explicitly that transition to an extended space model (with spinor structure) lead us to augmenting the number of basis wave functions of a quantum-mechanical scalar particle. Also, some possible manifestations of the extended space structure in matrix elements of physical quantities are discussed.

Parabolic cylindrical coordinates

These coordinates in the vector 3-space model are introduced by relations

$$x = \frac{u^2 - v^2}{2}\ ,\ y = u\,v\ ,\ z = z\ .v^2 = -x + \sqrt{x^2 + y^2}\ ,$$
$$u^2 = +x + \sqrt{x^2 + y^2}\ . \tag{1}$$

To cover all points of the vector space (x, y, z) it suffices any one from the following four solutions:

$$v = +\sqrt{-x + \sqrt{x^2 + y^2}}\ , \qquad u = \pm\sqrt{+x + \sqrt{x^2 + y^2}}\ , \tag{2}$$
$$v = -\sqrt{-x + \sqrt{x^2 + y^2}}\ , \qquad u = \pm\sqrt{+x + \sqrt{x^2 + y^2}}\ ,$$
$$v = \pm\sqrt{-x + \sqrt{x^2 + y^2}}\ , \qquad u = +\sqrt{+x + \sqrt{x^2 + y^2}}\ ,$$
$$v = \pm\sqrt{-x + \sqrt{x^2 + y^2}}\ , \qquad u = -\sqrt{+x + \sqrt{x^2 + y^2}}\ . \tag{3}$$

For definiteness, let us use the first variant from (2):

$$v = +\sqrt{-x + \sqrt{x^2 + y^2}}\ , \qquad u = \pm\sqrt{+x + \sqrt{x^2 + y^2}}\ .$$

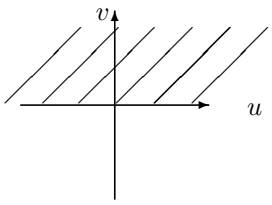

Fig 1. The domain $G(u, v)$ to parameterize the vector model

Correspondence between the points (x, y) and (u, v) can be illustrated by the formulas and schemes:

$$u = k\ \cos\phi\ , \qquad v = k\ \sin\phi\ , \qquad \phi \in [\ 0,\ \pi\ ]\ ;$$
$$x = (k^2/2)\ \cos 2\phi\ , \qquad y = (k^2/2)\ \sin 2\phi\ , \qquad 2\phi \in [0,\ 2\pi] \tag{4}$$

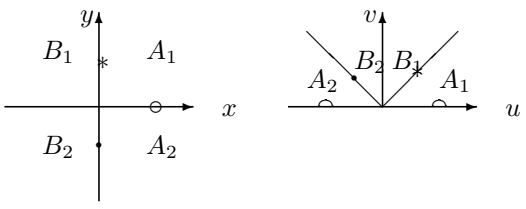

Fig 2. The mapping $G(x, y) \Longrightarrow G(u, v)$

In the following, when turning to the case of spinor space, we will see the complete symmetry between coordinates u v: namely, they are referred to Cartesian coordinates of the extended model $(x,y,z) \oplus (x,y,z)$ through the formulas

$$v = \pm\sqrt{-x + \sqrt{x^2+y^2}}\,, \qquad u = \pm\sqrt{+x + \sqrt{x^2+y^2}}\,. \tag{5}$$

the latter can be illustrated by the Fig 3:

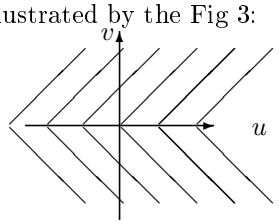

Fig 3. $\tilde{G}(u,v)$ **to cover spinor space**

The metric of 3-space in parabolic cylindrical coordinates is

$$dl^2 = dx^2 + dy^2 + dz^2 = (u^2+v^2)(du^2+dv^2) + dz^2\,.$$

Solutions of the Klein-Fock equation and functions of parabolic cylinder

Let us consider the Klein-Fock equation specified for cylindric parabolic coordinates:

$$\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{u^2+v^2}\,(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}) - \frac{m^2c^2}{\hbar^2} \right]\,\Psi = 0\,, \tag{6}$$

After separating the variables (t,z) from (u,v) by the substitution
$\Psi(t,u,v,\phi) = e^{-i\epsilon t/\hbar}\, e^{ipz/\hbar}\, U(u)\, V(v)$ one gets

$$\left[\frac{1}{U}\,\frac{d^2U}{du^2} + (\frac{\epsilon^2}{\hbar^2c^2} - \frac{m^2c^2}{\hbar^2} - \frac{p^2}{\hbar^2})\,u^2 \right]$$
$$+ \left[\frac{1}{V}\,\frac{d^2V}{dv^2} + (\frac{\epsilon^2}{\hbar^2c^2} - \frac{m^2c^2}{\hbar^2} - \frac{p^2}{\hbar^2})\,v^2 \right] = 0\,. \tag{7}$$

In the following, the notation is used

$$\lambda^2 = (\frac{\epsilon^2}{\hbar^2c^2} - \frac{m^2c^2}{\hbar^2} - \frac{p^2}{\hbar^2})\,, \qquad [\lambda] = \frac{1}{\text{meter}}\,.$$

Introducing two separation constants a and b $(a+b=0)$, from (7) we can derive two separate equations in variables u and v respectively:

$$\frac{d^2U}{du^2} + (\,\lambda^2\,u^2 - a\,)\,U = 0\,, \qquad \frac{d^2V}{dv^2} + (\,\lambda^2\,v^2 - b\,)\,V = 0\,. \tag{8}$$

Canonical form of differential equation of parabolic cylinder (type 2, [39]) is

$$\frac{d^2F}{d\xi^2} + (\,\frac{\xi^2}{4} - \alpha\,)\,F = 0\,. \tag{9}$$

Transition in equations ((8) to the canonical form is reached through the use of dimensionless variables)

$$\sqrt{2\lambda}\,u \;\to\; u\,, \qquad \frac{a}{2\lambda} \;\to\; a\,, \quad \sqrt{2\lambda}\,v \;\to\; v\,, \qquad \frac{b}{2\lambda} \;\to\; b\,. \tag{10}$$

So that equations (8) will take the form:

$$\frac{d^2U}{du^2} + \left(\frac{u^2}{4} - a\right) U = 0\,, \qquad \frac{d^2V}{dv^2} + \left(\frac{v^2}{4} - b\right) V = 0\,. \tag{11}$$

As known, solutions of equation (9) can be found as a series:

$$F(\xi) = c_0 + c_1\xi + c_2\xi^2 + \sum_{k=1,2,\ldots} c_{2k+1}\xi^{2k+1} + \sum_{k=1,2,\ldots} c_{2k+2}\xi^{2k+2}; \tag{12}$$

in (12) the terms of even and odd powers of ξ are distinguished. After substituting (12) into (9) we get:

$$\begin{aligned}
&\left[c_2 2 + \sum_{k=1,2,\ldots} c_{2k+1}(2k+1)(2k)\xi^{2k-1} + \right. \\
&\qquad \left. \sum_{k=1,2,\ldots} c_{2k+2}(2k+2)(2k+1)\xi^{2k}\right] + \\
&\frac{1}{4}\left[c_0\xi^2 + c_1\xi^3 + c_2\xi^4 + \sum_{k=1,2,\ldots} c_{2k+1}\xi^{2k+3} + \right. \\
&\qquad \left. \sum_{k=1,2,\ldots} c_{2k+2}\xi^{2k+4}\right] - \\
&-\alpha\left[c_0 + c_1\xi + c_2\xi^2 + \sum_{k=1,2,\ldots} c_{2k+1}\xi^{2k+1} + \right. \\
&\qquad \left. \sum_{k=1,2,\ldots} c_{2k+2}\xi^{2k+2}\right] = 0,
\end{aligned} \tag{13}$$

or separating terms of even and odd powers

$$\begin{aligned}
&\left[c_2 2 + \sum_{k=1,2,\ldots} c_{2k+2}(2k+2)(2k+1)\xi^{2k} + \frac{1}{4}c_0\xi^2 + \frac{1}{4}c_2\xi^4 + \right. \\
&\left. + \frac{1}{4}\sum_{k=1,2,\ldots} c_{2k+2}\xi^{2k+4} - \alpha c_0 - \alpha c_2\xi^2 - \alpha \sum_{k=1,2,\ldots} c_{2k+2}\xi^{2k+2}\right]_{\text{even}} + \\
&+ \left[\sum_{k=1,2,\ldots} c_{2k+1}(2k+1)(2k)\xi^{2k-1} + \frac{1}{4}c_1\xi^3 + \frac{1}{4}\sum_{k=1,2,\ldots} c_{2k+1}\xi^{2k+3} - \right. \\
&\qquad \left. - \alpha c_1\xi - \alpha \sum_{k=1,2,\ldots} c_{2k+1}\xi^{2k+1}\right]_{\text{odd}} = 0,
\end{aligned} \tag{14}$$

and further

$$\Big[\xi^0(2c_2-\alpha c_0)+\xi^2(c_4 4\times 3+\frac{c_0}{4}-\alpha c_2)+\xi^4(c_6 6\times 5+\frac{c_2}{4}-\alpha c_4)+ \\ +\sum_{k=3,4,...} c_{2k+2}(2k+2)(2k+1)\xi^{2k} \\ +\frac{1}{4}\sum_{k=1,2,...} c_{2k+2}\xi^{2k+4}-\alpha\sum_{k=2,3,...} c_{2k+2}\xi^{2k+2}\Big]_{\rm even}+ \\ \Big[\xi(c_3 3\times 2-\alpha c_1)+\xi^3(c_5 5\times 4+\frac{c_1}{4}-\alpha c_3)+ \\ +\sum_{k=3,4,...} c_{2k+1}(2k+1)(2k)\xi^{2k-1} \\ +\frac{1}{4}\sum_{k=1,2,...} c_{2k+1}\xi^{2k+3}-\alpha\sum_{k=2,3,...} c_{2k+1}\xi^{2k+1}\Big]_{\rm odd}=0. \qquad (15)$$

From this it follows

$$\Big[\xi^0(2c_2-\alpha c_0)+\xi^2(c_4 4\times 3+\frac{c_0}{4}-\alpha c_2)+\xi^4(c_6 6\times 5+\frac{c_2}{4}-\alpha c_4)+ \\ +\sum_{n=3,4,...}\left(c_{2n+2}(2n+2)(2n+1)+\frac{1}{4}c_{2n-2}-\alpha c_{2n}\right)\xi^{2n}\Big]_{\rm even}+ \\ \Big[\xi(c_3 3\times 2-\alpha c_1)+\xi^3(c_5 5\times 4+\frac{c_1}{4}-\alpha c_3)+ \\ +\sum_{n=3,4,...}\left(c_{2n+1}(2n+1)(2n)+\frac{1}{4}c_{2n-3}-\alpha c_{2n-1}\right)\xi^{2n-1}\Big]_{\rm odd}=0\,. \qquad (16)$$

Setting each coefficient at a ξ^k equal to zero one derives two independent groups of recurrent relations: <u>even</u>

$$\begin{aligned} \xi^0: &\qquad 2\,c_2-\alpha\,c_0=0\,,\\ \xi^2: &\qquad c_4\,4\times 3+\tfrac{c_0}{4}-\alpha\,c_2=0\,,\\ \xi^4: &\qquad c_6\,6\times 5+\tfrac{c_2}{4}-\alpha\,c_4=0\,,\\ n=3,4,...,\ \xi^{2n}: &\qquad c_{2n+2}(2n+2)(2n+1)+\tfrac{1}{4}\,c_{2n-2}-\alpha\,c_{2n}\ =0\,; \end{aligned} \qquad (17)$$

<u>odd</u>

$$\begin{aligned} \xi^1: &\qquad c_3\,3\times 2-\alpha\,c_1=0\,,\\ \xi^3: &\qquad c_5\,5\times 4+\tfrac{c_1}{4}-\alpha\,c_3=0\,,\\ n=3,4,...,\ \xi^{2n-1}: &\qquad c_{2n+1}(2n+1)(2n)+\tfrac{1}{4}\,c_{2n-3}-\alpha\,c_{2n-1}=0\,. \end{aligned} \qquad (18)$$

Taking into account the absence of any connection of equations (17) and (18) one can construct two linearly independent solutions (even and odd respectively):

<u>even</u>

$$F_1(\xi^2)=1+a_2\frac{\xi^2}{2!}+a_4\frac{\xi^4}{4!}+...,$$

$$a_2=\alpha\,,\qquad a_4=\alpha^2-\frac{1}{2}\,,\qquad c_6=\alpha^3-\frac{7}{2}\alpha\,,$$

$$n=3,4,...:\qquad a_{2n+2}=\alpha\,a_{2n}-\frac{(2n)(2n-1)}{4}\,a_{2n-2}\ ;$$

(19)

odd

$$F_2(\xi) = \xi + a_3 \frac{\xi^3}{3!} + a_5 \frac{\xi^5}{5!} + ... ,$$

$$a_3 = \alpha , \qquad a_5 = \alpha^2 - \frac{3}{2} ,$$

$$n = 3, 4, ... : \qquad a_{2n+1} = \alpha \, a_{2n-1} - \frac{(2n-1)(2n-2)}{4} \, a_{2n-3} \ .$$

(20)

The set of basis wave functions for Klein-Fock particle, the role and manifestation of vector and spinor space structures respectively

Having combined two previous solutions F_1 and F_2, we can obtain four types of the wave functions, solutions of the Klein-Fock equation in cylindrical parabolic coordinates (we will change the notation: $F_1 \Longrightarrow E$; $F_1 \Longrightarrow O$:

$$\begin{array}{ll} (\text{even} \otimes \text{even}) : & \Phi_{++} = E(a, u^2) \, E(-a, v^2), \\ (\text{odd} \otimes \text{odd}) : & \Phi_{--} = O(a, u) \, O(-a, v \,) \ , \\ (\text{even} \otimes \text{odd}) : & \Phi_{+-} = E(a, u^2) \, O(-a, v \,) \ , \\ (\text{odd} \otimes \text{even}) : & \Phi_{-+} = O(a, u) \, E(-a, v^2) \ . \end{array} \tag{21}$$

Having in mind relation between (u, v) and (x, y), one readily notes behavior of the wave functions constructed at the point $x = 0, y = 0$ (variable z is omitted):

$$\begin{array}{ll} (\text{even} \otimes \text{even}) : & \Psi_{++}(x = 0, y = 0) \neq 0 \ , \\ (\text{odd} \otimes \text{odd}) : & \Psi_{--}(x = 0, y = 0) = 0 \ , \\ (\text{even} \otimes \text{odd}) : & \Psi_{+-}(x > 0, y = 0) = 0 \ , \\ (\text{odd} \otimes \text{even}) : & \Psi_{-+}(x < 0, y = 0) = 0 \ . \end{array} \tag{22}$$

Now let us consider which restrictions for the wave functions Ψ are imposed by the requirement of single-valued ness. Two peculiarities in parameterizing are substantial:

$$\underline{v = 0 \ :} \qquad x = +\frac{u^2}{2} \geq 0, \ y = 0; \qquad \underline{u = 0 \ :} \qquad x = -\frac{v^2}{2} \leq 0, \ y = 0 \ . \tag{23}$$

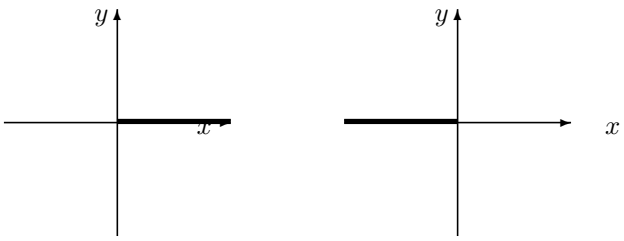

Fig 4. The peculiarities in parametrization

The above four solutions (21) behave in peculiar regions as follows: $\underline{(\text{even} \otimes \text{even}) :}$

$$\Phi_{++}(a;u=0,v)=E(a,u^2=0)E(-a,v^2)=$$
$$E(-a,v^2)=+\,\Phi_{++}(a;u=0,-v)\ ,$$
$$\Phi_{++}(a;+u,v=0)=E(a,u^2)\ E(-a,v^2=0)=$$
$$E(+a,u^2)=+\,\Phi_{++}(a;-u,v=0)\ ,$$

(odd $\otimes$ odd) :

$$\Phi_{--}(a;u=0,+v)=O(a,u=0)\ O(-a,v\ )=+\,\Phi_{--}(a;u=0,-v)=0$$
$$\Phi_{--}(a;u,v=0)=O(+a,u)\ )O(-a,v=0\ )=+\,\Phi_{--}(a;-u,v=0)=0\ ,$$

(even $\otimes$ odd) :

$$\Phi_{+-}(a;u=0,+v)=E(a,u^2=0)\ O(-a,v\ )=$$
$$O(-a,v\ )=-\,O(-a,-v\ )=-\,\Phi_{+-}(a;u=0,-v)\ ,$$
$$\Phi_{+-}(a;u,v=0)=E(+a,u^2)\ O(-a,v=0\ )=\Phi_{+-}(a;-u,v=0)=0\ ,$$

(odd $\otimes$ even) :

$$\Phi_{-+}(a;u=0,+v)=O(+a,u=0)E(-a,v^2)=\Phi_{-+}(a;u=0,-v)=0\ ,$$
$$\Phi_{-+}(a;+u,v=0)=O(+a,u)\ E(-a,v^2=0)=$$
$$O(+a,u)=-O(a,-u)=-\,\Phi_{-+}(a;-u,v=0)\ .$$

(24)

Taking in mind the Fig.1 and the Fig. 3, one cam immediately conclude: solutions Φ of the types $(++)$ and $(--)$ are single-valued in the space with vector structure, whereas the solutions of the types $(+-)$ and $(-+)$ are not single-valued in space with vector structure, so these types $(+-)$ and $(-+)$ must be rejected. However, these solutions $(+-)$ and $(-+)$ must be retained in the space with spinor structure.

That dividing of the basis wave functions into two subsets may be formalized mathematically with the help of special discrete operator acting in spinor space:

$$\hat{\delta}=\left|\begin{array}{cc}-1&0\\0&-1\end{array}\right|,\qquad \hat{\delta}\left|\begin{array}{c}u\\v\end{array}\right|=\left|\begin{array}{c}-u\\-v\end{array}\right|\ . \tag{25}$$

It is easily verified that solutions single-valued in the vector space model are eigenfunctions of δ with eigenvalue $\delta=+1$:

$$\hat{\delta}\ \Phi_{++}(a;u,v)=+\,\Phi_{++}(a;u,v)\ ,\ \ \hat{\delta}\ \Phi_{--}(a;u,v)=+\,\Phi_{--}(a;u,v)\ , \tag{26}$$

and additional ones acceptable only in the spinor space model, are eigenfunction with the eigenvalue $\delta=-1$:

$$\hat{\delta}\ \Phi_{+-}(a;u,v)=-\,\Phi_{+-}(a;u,v)\ ,\ \ \hat{\delta}\ \Phi_{-+}(a;u,v)=-\,\Phi_{-+}(a;u,v)\ . \tag{27}$$

When using the spinor space model, two set (u,v) and $(-u,-v)$ represent different geometrical points in the spinor space, so the requirement of single valuedness as applied in the case of spinor space does not presuppose that the values of the wale functions must be equal in the points (u,v) and $(-u,-v)$:

$$\Phi(u,v)=\Phi((x,y)^{(1)})\neq\Phi(-u,-v)=\Phi((x,y)^{(2)})\ . \tag{28}$$

Now let us add some details more. In general, the vector plane (x,y) allows three inversion operations to which one can relate six discrete operations in spinor "plane" (u,v):

$$(x,y)\Longrightarrow(x,-y),\qquad \hat{\pi}=\left|\begin{array}{cc}+1&0\\0&-1\end{array}\right|,\qquad \hat{\pi'}=\hat{\delta}\ \hat{\pi}=-\hat{\pi}\ ,$$
$$(x,y)\Longrightarrow(-x,y),\qquad \hat{\omega}=\left|\begin{array}{cc}0&+1\\+1&0\end{array}\right|,\qquad \hat{\omega'}=\hat{\delta}\ \hat{\omega}=-\hat{\omega}\ ,$$
$$(x,y)\Longrightarrow(-x,-y),\qquad \hat{R}=\left|\begin{array}{cc}0&-1\\+1&0\end{array}\right|,\qquad \hat{R'}=\hat{\delta}\ \hat{R}=-\hat{R}\ . \tag{29}$$

One can easily construct eigenfunctions of these discrete operations (29) as well. For instance, let us consider the operator $\hat{R} = \hat{\omega}\,\hat{\pi}$. Noting two identities

$$\hat{R}\Phi_{++}(a;u,v) = \hat{R}E(a,u^2)E(-a,v^2) = E(a,v^2)E(-a,u^2) = \Phi_{++}(-a;u,v), \tag{30}$$

$$\hat{R}\Phi_{--}(a;u,v) = \hat{R}O(a,u)O(-a,v) = O(a,-v)O(-a,u) = -\Phi_{--}(-a;u,v). \tag{31}$$

one can easily construct the eigen-functions of the operator $\hat{R}$ (arguments are omitted):

$$\Phi_{++}^{(R=\pm 1)} = \Phi_{++}(a) \pm \Phi_{++}(-a)\,, \quad \hat{R}\ \Phi_{++}^{(R=\pm 1)} = \pm\,\Phi_{++}^{(R=\pm 1)}\,; \tag{32}$$

$$\Phi_{--}^{(R=\pm 1)} = \Phi_{--}(a) \mp \Phi_{--}(-a)\,, \quad \hat{R}\,\Phi_{--}^{(R=\pm 1)} = \pm\,\Phi_{--}^{(R=\pm 1)}\,. \tag{33}$$

In the same way, taking into account the identities

$$\hat{R}\,\Phi_{+-}(a;u,v) = \hat{R}\,E(a,u^2)\,O(-a,v) = E(a,v^2)\,O(-a,u) = +\,\Phi_{-+}(-a;u,v), \tag{34}$$

$$\hat{R}\,\Phi_{-+}(a;u,v) = \hat{R}\,O(a,u)\,E(-a,v^2) = O(a,-v)\,E(-a,u^2) = -\,\Phi_{+-}(-a;u,v)\,, \tag{35}$$

one can easily construct eigenfunctions with complex eigenvalues:

$$\varphi^{(R=\mp i)} = \Phi_{+-}(a) \pm i\,\Phi_{-+}(-a)\,, \quad \hat{R}\,\varphi^{(R=\pm i)} = \pm\,i\,\varphi^{(R=\pm i)}\,;$$

(36)

$$\varphi^{(R=\mp i)}(-a) = \Phi_{+-}(-a) \pm i\,\Phi_{-+}(+a)\,, \quad \hat{R}\,\varphi^{(\pm i)}(-a) = \pm\,i\,\varphi^{(\mp i)}(-a)\,.$$

(37)

Thus, there exist quite a definite classification of the Klein-Fock solutions in cylindrical parabolic coordinates in terms of quantum numbers, eigenvalues of the following operator (an explicit form $\hat{A}$ will be given below)

$$i\frac{\partial}{\partial t} \Longrightarrow \epsilon\,, \quad -i\frac{\partial}{\partial z} \Longrightarrow p\,, \qquad \hat{A} \Longrightarrow a\,,\ (\hat{\delta},\hat{R}) \Longrightarrow (\delta = \pm 1, R = \pm 1)\,.$$

(38)

As a base to classify solutions of the Klein-Fock equation, instead of $(\hat{\delta},\hat{R})$ one might have taken other two operator: for instance, $\hat{\delta}$ and $\hat{\omega}$. Then, allowing for the identities

$$\hat{\omega}\ \Phi_{++}(a;u,v) = \hat{\omega}\ \ E(a,u^2)\ E(-a,v^2) = E(a,v^2)\ E(-a,u^2) = \Phi_{++}(-a;u,v)\,, \tag{39}$$

$$\hat{\omega}\ \Phi_{--}(a;u,v) = \hat{\omega}\ \ O(a,u)\ O(-a,v) = O(a,v)\ O(-a,u) = \Phi_{--}(-a;u,v)\,. \tag{40}$$

We can construct eigenfunctions of the operator $\hat{\omega}$:

$$\Phi_{++}^{(\omega=\pm 1)} = \Phi_{++}(a) \pm \Phi_{++}(-a)\,, \quad \hat{\omega}\ \Phi_{++}^{(\omega=\pm 1)} = \pm\,\Phi_{++}^{(\omega=\pm 1)}\,; \tag{41}$$

$$\Phi_{--}^{(\omega=\pm 1)} = \Phi_{--}(a) \pm \Phi_{--}(-a)\,, \quad \hat{\omega}\,\Phi_{--}^{(\omega=\pm 1)} = \pm\,\Phi_{--}^{(\omega=\pm 1)})\,. \tag{42}$$

In the same manner, for additional solutions we have

$$\hat{\omega}\ \Phi_{+-}(a;u,v) = \hat{\omega}\ E(a,u^2)\ O(-a,v) = E(a,v^2)\ O(-a,u) = +\ \Phi_{-+}(-a;u,v), \tag{43}$$

$$\hat{\omega}\ \Phi_{-+}(a;u,v) = \hat{\omega}\ O(a,u)\ E(-a,v^2) = O(a,v)\ E(-a,u^2) = \ \Phi_{+-}(-a;u,v)\ , \tag{44}$$

therefore, the eigenfunctions may be given as

$$\varphi^{(\omega=\pm 1)} = \Phi_{+-}(a) \pm\ \Phi_{-+}(-a)\ ,\ \ \hat{\omega}\ \varphi^{(\omega=\pm 1)} = \pm\ \varphi^{(\omega=\pm 1)}\ ; \tag{45}$$

$$\varphi^{(\omega=\pm 1)} = \Phi_{+-}(-a) \pm\ \Phi_{-+}(+a)\ ,\ \ \hat{\omega}\ \varphi^{(\omega=\pm 1)} = \pm\ \varphi^{(\omega=\mp 1)}\ ; \tag{46}$$

It is easy to obtain some classifications with the help of $(\hat{\delta},\ \hat{\pi})$. Indeed,

$$\begin{aligned}
\hat{\pi}\ \Psi_{++}(a;u,v) &= \hat{\pi}\ F_1(a,u^2)\ F_1(-a,v^2) = \\
F_1(a,u^2)\ F_1(-a,v^2) &= +\ \Psi_{++}(a;u,v)\ , \\
\hat{\pi}\ \Psi_{--}(a;u,v) &= \hat{\pi}\ F_2(a,u)\ F_2(-a,v) = \\
F_2(a,-u)\ F_2(-a,v) &= -\ \Psi_{--}(a;u,v)\ , \\
\hat{\pi}\ \Psi_{+-}(a;u,v) &= \hat{\pi}\ F_1(a,u^2)\ F_2(-a,v) = \\
F_1(a,u^2)\ F_2(-a,v) &= +\ \Psi_{+-}(a;u,v)\ , \\
\hat{\pi}\ \Psi_{-+}(a;u,v) &= \hat{\pi}\ F_2(a,u)\ F_1(-a,v^2) = \\
F_2(a,-u)\ F_1(-a,v^2) &= -\ \Psi_{-+}(a;u,v)\ .
\end{aligned} \tag{47}$$

Remembering eqs. (25) – (28), one can conclude that the basic solutions are eigenfunctions of two discrete operators $\hat{\delta}$ and $\hat{\pi}$.

Boundary properties of the wave functions constructed can be illustrated by the schemes:

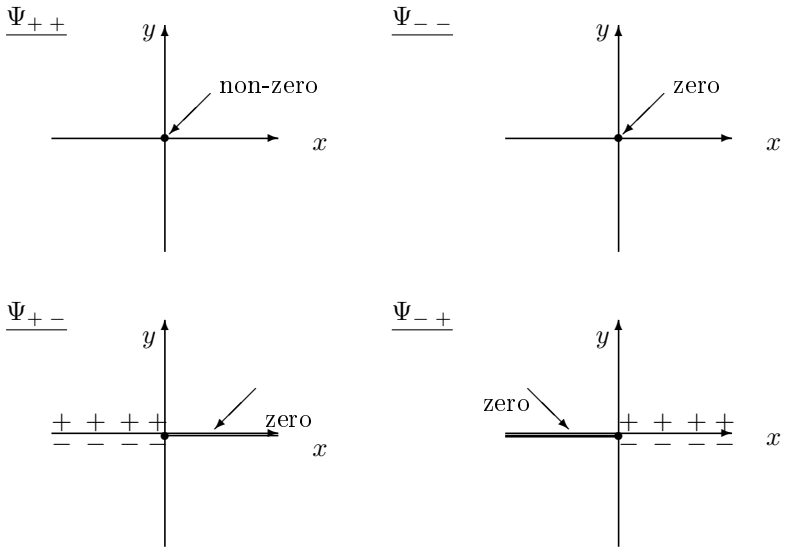

Fig 7. Boundary behavior of the wave functions in (x,y)-plane

Orthogonality and completeness of the bases for vector and spinor space models

Now let us consider the scalar multiplication

$$\int \Psi^*_{\mu'}\ \Psi_{\mu}\ \sqrt{-g}\ dtdzdudv\ . \tag{48}$$

of the basic wave functions constructed:

$$\begin{aligned}
\Psi_{++}(\epsilon,p,a) &= e^{i\epsilon t}\, e^{ipz}\, \Phi_{++}(a;u,v) = e^{i\epsilon t}\, e^{ipz}\, E(+a,u^2)\, E(-a,v^2)\ , \\
\Psi_{--}(\epsilon,p,a) &= e^{i\epsilon t}\, e^{ipz}\, \Phi_{--}(a;u,v) = e^{i\epsilon t}\, e^{ipz}\, O(+a,u)\, O(-a,v\ )\ , \\
\Psi_{+-}(\epsilon,p,a) &= e^{i\epsilon t}\, e^{ipz}\, \Phi_{+-}(a;u,v) = e^{i\epsilon t}\, e^{ipz}\, E(+a,u^2)\, O(-a,v\ )\ , \\
\Psi_{-+}(\epsilon,p,a) &= e^{i\epsilon t}\, e^{ipz}\, \Phi_{-+}(a;u,v) = e^{i\epsilon t}\, e^{ipz}\, O(+a,u)\, E(-a,v^2)\ .
\end{aligned} \tag{49}$$

μ and μ' stand for generalized quantum numbers. In the first place, interesting integrals are (arguments (a;u,v) are omitted):

in vector space

$$I_0 = \int_0^{+\infty} dv \int_{-\infty}^{+\infty} du\ \Phi^*_{++}\ \Phi_{--}\ (u^2+v^2), \tag{50}$$

in spinor space

$$\begin{aligned}
I_1 &= \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\ \Phi^*_{++}\ \Phi_{--}\ (u^2+v^2)\ , \\
I_2 &= \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\ \Phi^*_{+-}\ \Phi_{-+}\ (u^2+v^2)\ , \\
I_3 &= \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\ \Phi^*_{++}\ \Phi_{+-}\ (u^2+v^2)\ , \\
I_4 &= \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\ \Phi^*_{++}\ \Phi_{-+}\ (u^2+v^2)\ , \\
I_5 &= \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\ \Phi^*_{--}\ \Phi_{+-}\ (u^2+v^2)\ , \\
I_6 &= \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\ \Phi^*_{--}\ \Phi_{-+})\ (u^2+v^2)\ .
\end{aligned} \tag{51}$$

Integral I_0 in vector space vanishes identically

$$\begin{aligned}
I = \int_0^{+\infty} dv \int_{-\infty}^{+\infty} du E(+a,u^2)E(-a,v^2) \times \\
O(+a,u)O(-a,v)(u^2+v^2) = \\
= \int_0^{+\infty} dv \int_{-\infty}^{+\infty} du E(+a,u^2)\underline{O(+a,u)} \times \\
E(-a,v^2); O(-a,v)(u^2+v^2) \equiv 0,
\end{aligned}$$

because integration in variable $u \in (-\infty,+\infty)$ is done for an odd function of u in symmetrical region $u \in (-\infty,+\infty)$. By the same reasons, integral I_1 in spinor space vanishes as well.

The integral I_2 vanishes

$$\begin{aligned}
I_2 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\ \ E(+a,u^2)\ O(-a,v\ ) \\
O(+a,u)\ E(-a,v^2)\ (u^2+v^2) \equiv 0\ ,
\end{aligned}$$

because integration is done for an odd function in v,u-variables, in symmetrical regions $v \in (-\infty,+\infty)$ and $u \in (-\infty,+\infty)$. Integral I_3 vanishes

$$\begin{aligned}
I_3 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\ E(+a,u^2)\ E(-a,v^2) \\
E(+a,u^2)\ O(-a,v\ )\ u^2+v^2) \equiv 0\ ,
\end{aligned}$$

because integration is done for odd function of v variable, in the symmetrical region $v \in (-\infty, +\infty)$. Integral I_4 vanishes

$$I_4 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\; E(+a, u^2)\; E(-a, v^2)$$
$$O(+a, u)\; E(-a, v^2)\; (u^2 + v^2) \equiv 0\;,$$

because integration is done for an odd function of U in symmetrical region $u \in (-\infty, +\infty)$. Integral I_5 vanishes

$$I_5 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\; O(+a, u)\; O(-a, v\;)$$
$$E(+a, u^2)\; O(-a, v\;)\; (u^2 + v^2) \equiv 0\;,$$

because one integrates an odd function of u in symmetrical region $u \in (-\infty, +\infty)$. Integral I_6 vanishes

$$I_6 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\; O(+a, u)\; O(-a, v\;)$$
$$O(+a, u)\; E(-a, v^2)\; (u^2 + v^2) \equiv 0\;,$$

because one integrates an odd function of v in symmetrical region $v \in (-\infty, +\infty)$.

Thus, vanishing integrates $I_0, I_1 ... I_6$ from (50),(51) shows that the formulas (49) provide us with orthogonal basis for Hilbert space $\Psi(unv, z)$, where (u, v, z) belong to an extended (spinor) space model.

On matrix elements of physical observables, in vector and spinor space models

The question of principle is how transition from vector to spinor space model can influence result of calculation of matrix elements for physical quantities. As an example, let us consider matrix elements for operator of coordinates: One may calculate matrix elements of basic initial coordinates u, v or there 2-order derivative coordinates x, y:

$$(u, v) \qquad \text{or} \qquad x = \frac{u^2 - v^2}{2}\;,\quad y = uv\;. \tag{52}$$

With the use of the above rules – integral for an odd function in symmetrical region vanishes identically – one can derive simple section rules for matrix elements (for simplicity we restrict ourselves only to the degeneracy in discrete quantum number $++, --, +-, -+$ taking ϵ, p, a fixed):

in vector space

$x_{\mu',\mu}$	$++$	$--$
$++$	$\neq 0$	0
$--$	0	$\neq 0$

,

$y_{\mu',\mu}$	$++$	$--$
$++$	0	$\neq 0$
$--$	$\neq 0$	0

in spinor space

$x_{\mu',\mu}$	$++$	$--$	$+-$	$-+$
$++$	$\neq 0$	0	0	0
$--$	0	$\neq 0$	0	0
$+-$	0	0	$\neq 0$	0
$-+$	0	0	0	$\neq 0$

,

$y_{\mu',\mu}$	$++$	$--$	$+-$	$-+$
$++$	0	$\neq 0$	0	0
$--$	$\neq 0$	0	0	0
$+-$	0	0	0	$\neq 0$
$-+$	0	0	$\neq 0$	0

The same for coordinates u and v looks: in vector space

$u_{\mu',\mu}$	$++$	$--$
$++$	0	$\neq 0$
$--$	$\neq 0$	0

,,

$v_{\mu',\mu}$	$++$	$--$
$++$	$\neq 0$	0
$--$	0	$\neq 0$

in spinor space

$u_{\mu',\mu}$	$++$	$--$	$+-$	$-+$
$++$	0	0	0	$\neq 0$
$--$	0	0	$\neq 0$	0
$+-$	0	$\neq 0$	0	0
$-+$	$\neq 0$	0	0	0

,

$v_{\mu',\mu}$	$++$	$--$	$+-$	$-+$
$++$	0	0	$\neq 0$	0
$--$	0	0	0	$\neq 0$
$+-$	$\neq 0$	0	0	0
$-+$	0	$\neq 0$	0	0

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Relativistic motion in nanostructures: interesting details by a new Drude-Lorentz-like model

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Abstract: *In this paper it is presented an interesting overview concerning the theoretical efforts in the understanding of phenomena at nanoscale; starting by the Drude model we arrive to the last appeared Drude-Lorentz-like models, in the case of relativistic motion of carriers in nano-bio-structures. After this, it will focused in particular about the analysis of the velocities correlation function of a new appeared model, which has a wide scale range of applicability; in this contest the nanoscale will be considered. The theoretical framework is performed, so as examples of application.*

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Introduction

The charge transport is one of the most important aspects at nanoscale; it can be influenced by particles dimensions and presents different characteristics with respect to those of bulk. In mesoscopic systems the mean free path of charges, related to scattering phenomena, can become larger than the particle dimensions; the transport depends therefore by dimensions and in principle corrections of the transport bulk theories are possible, for considering this phenomenon. Also in a thin film, the smallest nanostructure dimension can be less than the free displacement, requiring so modifications to existing theoretical transport models. At theoretical level, various techniques have been used for the knowledge of transport phenomena, in particular analytical descriptions based on transport equations and numerical approaches, as classical and quantum Monte Carlo simulations.

Recently it has appeared a new theoretical approach, based on the complete Fourier transform of the frequency-dependent complex conductivity of the studied system [1,2]. With this approach it is possible to calculate exactly the analytical expressions of the most important transport functions, i.e. the velocities correlation function $< \vec{v}(t) \cdot \vec{v}(0) >_T$, the mean square deviation of position $R^2(t)$ and the diffusion coefficient $D(t) = 1/2\,(dR^2(t)/dt)$.

One of the most important experimental technique for the study of the frequency-dependent complex-valued far-infrared photoconductivity $\sigma(\omega)$ is the Time-resolved THz Spectroscopy (TRTS), an ultrafast non-contact optical probe; data are usually fitted via Drude-Lorentz, Drude-Smith [3] and Effective Medium Models [4].

An interesting overview of the fundamental utilized models will be illustrated; starting by the Drude model, we consider its extensions, arriving to new results by a recently appeared Drude-Lorentz-like model, which involves the quantum-relativistic reality.

Past and recent considered models

In relation to metals, scientists thought in terms of models in which the electrons are relatively free and can move under the influence of electric fields. Historically two models of the elementary metals theory were born:

a) the "Drude model", published in 1900 and based on the kinetic theory of an electron gas in a solid [5,6]. It was assumed that electrons have the same average kinetic energy E_m;

b) a variation of the Drude model, integrated with the foundations of quantum mechanics, called "Sommerfeld model" [7].

In the Drude model the assumption of a mechanism of collisions among ions and electrons allows the thermal equilibrium for the electrons, implying the application of the kinetic theory of gases. The free electrons have only kinetic energy, therefore the average energy is $E_m = (3/2)\,k_B\,T$, where k_B is the Boltzmann's constant. The correlation with an average quadratic velocity v_m is possible through the relation $E_m = (3/2)\,k_B\,T = m\,v_m^2/2$, where m is the free electron mass. At environment temperature is v_m of order of 10^7 cm/s and represents the average thermal electrons velocity. It was assumed also that the time of diffraction is very small with respect to every other considered time. Through such collisions, electrons acquire a thermal equilibrium corresponding to the temperature T of the metal. The possible presence of a constant electric field determines an extra average velocity (the "drift velocity") given by $v_d = -(e\,E/m)\,t$. The relaxation time τ is defined as the average time between two collisions, getting a mean free path $l_{mfp} = v_m\,\tau$. The current density is $\vec{J} = \sigma_{cond}\,\vec{E}$, where σ_{cond} is the electric conductivity. This result has been an important goal of the classic theory in relation to the metals conduction.

The Lorentz model, published in 1905, is a refining of the Drude model containing statistical aspects [8]. Electrons are considered as free charges, with charge "-e" and described by a maxwellian velocity distribution. An electron gas in a spatial region with a constant electric field has a constant drift velocity, corresponding to a current density $\vec{J}$, which is proportional to the applied field $\vec{J} = \sigma_0\,\vec{E}$, with $\sigma_0 = n\,e^2\,\tau\,/\,m$ (n is the electron density). Estimating the relaxation time τ, Drude and Lorentz obtained conductivity values in good accordance with experiments. In presence of an electric field of the form $E\,(t) = E_0\,e^{-\,i\,\omega\,t}$, the complex conductibility is writable as $\sigma_\omega = \sigma_0\,/\,(1 - i\,\omega\,\tau)$. Such model, known as "Drude-Lorentz model", received some success, but has also underlined serious difficulties.

Starting by the Drude-Lorentz model, it is possible to obtain the velocities correlation function, the quadratic average distance crossed by the charges as a function of time and examine directly the possible compatibility with the Einstein relation $D/\mu = k_B\,T\,/\,e$, where D is the diffusion coefficient and μ the mobility [8].

Considerable variations of the Drude-Lorentz model were made in the following years; the most used are:

c) the "Maxwell-Garnett model" (MG): in this model the dielectric function is given by a Drude term with an additional "vibrational" contribution at a finite frequency ω_o, which leads to a dielectric function of the form:

$$\varepsilon_{//}(\omega) = 1 - \frac{\omega_p^2}{\omega\,(\omega + i/\tau)} + \frac{\omega_s^2}{\omega_0^2 - \omega^2 - i\,\gamma\,\omega} \tag{1}$$

where the amplitude ω_s, the resonant frequency ω_0 and the damping constant γ are material-dependent constants. The MG model is used for describing an isotropic matrix, which contains spherical isolated inclusions, such as metal particles dispersed in a surrounding host matrix [9,10].

d) The "Effective Medium Theories" (EMTs): in this case the electromagnetic interactions between pure materials and host matrices are approximately taken into account. The commonly used EMTs include the MG model and the "Bruggeman model" (BR), particular variation of the MG model [11,12].

In the THz regime, the dielectric function $\varepsilon_m(\omega)$ consists normally of contributions of the high-frequency dielectric constant, conduction free electrons and lattice vibration:

$$\varepsilon_m(\omega) = \varepsilon_\infty - \frac{\omega_p^2}{\omega^2 + i\,\gamma\,\omega} + \sum_j \frac{\varepsilon_{st_j}\,\omega_{TO_j}^2}{\omega_{TO_j}^2 - \omega^2 - i\,\Gamma_j\,\omega} \tag{2}$$

In Eq. (2) ε_∞ is the high-frequency dielectric constant, the second term describes the contribution of free electrons or plasmons and the last term is related to optical phonons.

If the response originated mainly by the contribution of free electrons or plasmons, it is usually adopted the Drude model, in the form:

$$\varepsilon_m(\omega) = \varepsilon_\infty - \frac{\omega_p^2}{\omega^2 + i\,\gamma\,\omega} \tag{3}$$

describing with good approximation the dielectric properties of metals and semiconductors. If the interaction of a radiation field with the fundamental lattice vibration plays a dominant role and results in absorption of electromagnetic wave, due to the creation or annihilation of lattice vibration, the dielectric function $\varepsilon_m(\omega)$

mainly consists of the contributions of the lattice vibrations, expressed by the classical pseudo-harmonic phonon model in the first approximation:

$$\varepsilon_m(\omega) \,=\, \varepsilon_\infty \,+\, \frac{\varepsilon_{st}\,\omega_{TO}^2}{\omega_{TO}^2 \,-\, \omega^2 \,-\, i\,\gamma\,\omega} \tag{4}$$

e) The "Smith model": Smith started by the response theory for the optical conductivity, considering an electric field impulse applied to a system, in order to examine the answer with respect to the current [3,13]. The real part of $\sigma\,(\omega)$ results:

$$\int_0^\infty Re\,\sigma\,(\omega)\,d\omega \,=\, \frac{\pi}{2}\,j\,(0) \,=\, \frac{\omega_p^2}{8} \tag{5}$$

If the initial current decays exponentially to its initial value with relaxation time τ, it is possible to write:

$$j\,(t)/j\,(0) \,=\, \exp\,(\,-\,t\,/\,\tau\,) \tag{6}$$

from which the standard Drude formula is obtainable:

$$\sigma\,(\omega) \,=\, (n\,e^2\,\tau\,/\,m)\,/\,(1 \,-\, i\,\omega\,\tau) \tag{7}$$

Eq. (7) can be considered as the first term of a series of the form:

$$j\,(t)/j\,(0) \,=\, \exp\,(\,-\,t\,/\,\tau\,)\,\left[1 \,+\, \sum_{n\,=\,1}^{\infty} c_n\,\left(t/\tau\right)^n\,/\,n\,!\right] \tag{8}$$

The c_n factors hold into account of the original electrons velocity, remained after the n-th collision. The analytical form of the complex conductivity is:

$$\sigma\,(\omega) \,=\, \frac{n\,e^2\,\tau}{m\,(1 \,-\, i\,\omega\,\tau)}\,\left[1 \,+\, \sum_{n\,=\,1}^{\infty} \frac{c_n}{(1 \,-\, i\,\omega\,\tau)^n}\right] \tag{9}$$

A new appeared interesting model

A recent theoretical analytical formulation showed to fit very well with experimental scientific data and offers interesting new predictions of various peculiarities in nanostructures [1,2,14-21]. The model contains a gauge factor, which permits its use to study the dynamics of reality processes by sub-nanolevel to macrolevel, presenting oscillations in time, so as diffusivity characteristics in time [22].

The model is based on the complete Fourier transform of the frequency-dependent complex conductivity $\sigma\,(\omega)$ of the system, which can be deduced from linear response theory (Green-Kubo formula) [23,24]:

$$\sigma_{\beta\alpha}(\omega) = \,(e^2/\hbar V)\int_0^\infty dt e^{i\omega t}\int_0^\beta d\lambda < \vec{v}^{\alpha}(t - i\lambda)\vec{v}^{\beta}(0) > \tag{10}$$

By inversion of Eq. (10) it is possible to find the velocities correlation function $<\,\vec{v}\,(t)\,\cdot\,\vec{v}\,(0)\,>_T$ inside the integral. The presence of an integration from 0 to ∞ is however a problem for the analytical inversion, but it can be overcame evaluating the integral on the entire time axis (- ∞, + ∞). Considering the real part of the complex conductivity in Eq. (10), the extension to the entire time axis is possible and a complete Fourier transform can be performed, obtaining directly real velocities. The integral can be resolved in the complex plane considering a Cauchy integration; the velocities correlation function is evaluated exactly by the residue theorem [25]. With $<\,\vec{v}\,(t)\,\cdot\,\vec{v}\,(0)\,>_T$ it is possible to obtain the analytical form of $R^2(t)$ and $D(t)$.

The real part of $\sigma\,(\omega)$ results:

$$Re\,\sigma_{\beta\alpha}(\omega) = \frac{e^2}{2Vk_BT}\int_{-\infty}^{+\infty} dt\,\left\langle \vec{v}^{\,\alpha}(0)\vec{v}^{\,\beta}(t)\right\rangle_T e^{-i\omega t} \tag{11}$$

The integral in Eq. (11) spans the entire t-axis, so we can perform the complete inverse Fourier transform. It gives:

$$< \vec{v}^{\alpha}(0)\ \vec{v}^{\beta}(t) >_T = \frac{k_B\,T\,V}{\pi\,e^2} \int_{-\infty}^{+\infty} d\omega\ Re\,\sigma_{\beta\alpha}(\omega)\,e^{i\,\omega\,t} \tag{12}$$

The new introduced key idea is the possibility to perform a complete inversion of Eq. (12) on temporal scale, i.e. considering the entire time axis (-∞, +∞), not the half time axis (0, +∞), as usually considered in literature [26]. It has performed the classical and the quantum version of the indicated model [1,2,14]; currently the focus is on the relativistic version, of which new results are presented below.

Relativistic motion in nanostructures

The starting point is the motion equation of a particle travelling in a nanostructure. If we consider relativistic velocity, the considered dynamics law is:

$$\frac{d}{dt}\,(m_{part}\,\vec{v}) = \sum_i \vec{F}_i \tag{13}$$

We choose for simplicity the motion along the x-axis; about the forces acting on particle, it has been considered an outer passive elastic-type force of the form $F_{el} = K\,x$, a passive friction-type force of the form $F_{fr} = \lambda\,\dot{x}$, depending by velocity and with $\lambda = m_{part}/\tau$, and an outer oscillating electric field $E = e\,E_0\,e^{-\,i\,\omega\,t}$.

After analytical calculation, Eq. (13) becomes:

$$m_0\,a\,\gamma\,\left(1 + (\beta\,\gamma)^2\right) = -\,k\,x - \lambda\,\dot{x} + e\,E_0\,e^{-\,i\,\omega\,t} \tag{14}$$

with $\beta = v/c$ and $\gamma = 1\Big/\sqrt{1-\beta^2}$.

Considering for Eq. (14) solutions of the form:

$$x = x_0\,e^{-\,i\,\omega\,t} \tag{15}$$

the real part of the complex conductivity results:

$$Re\,\sigma = \frac{N\,e^2}{m_0\,\tau}\left(\frac{\omega^2}{\frac{\omega^2\,\gamma^2}{\tau^2} + (\omega_0^2 - \gamma\,(1+\beta^2\,\gamma^2)\,\omega^2)^2}\right) \tag{16}$$

With the procedure used in the classical case [2], searching values of ω which vanish the denominator of Eq. (16), it obtains solutions of the form:

$$\omega_{(R/I)_{rel}} = \frac{1}{2\,\tau\,\rho}\left(i \pm \sqrt{\frac{4\,\rho\,\omega_0^2\,\tau^2}{\gamma} - 1}\right) \tag{17}$$

with $\rho = 1 + \beta^2\,\gamma^2 = \gamma^2$. We have three cases in relation to the sign of the quantity $\Delta = \frac{4\,\rho\,\omega_0^2\,\tau^2}{\gamma} - 1$: $\Delta_{rel} > 0$, $\Delta_{rel} = 0$, $\Delta_{rel} < 0$.

The velocities correlation function $\langle \vec{v}(0)\cdot\vec{v}(t)\rangle$ has the following analytical form:

$\Delta_{rel} > 0$

$$\langle \vec{v}(0)\cdot\vec{v}(t)\rangle = \left(\frac{k_B\,T}{m_0}\right)\left(\frac{1}{\gamma^2\,\rho}\right)\,\exp\left(-\frac{t}{2\,\tau\,\rho}\right) \left(\cos\left(\frac{\alpha_{R_{rel}}}{2\,\rho}\,\frac{t}{\tau}\right) - \frac{1}{\alpha_{R_{rel}}}\,\sin\left(\frac{\alpha_{R_{rel}}}{2\,\rho}\,\frac{t}{\tau}\right)\right) \tag{18}$$

with $\alpha_{R_{rel}} = \sqrt{\frac{4\,\rho\,\omega_0^2\,\tau^2}{\gamma} - 1}$;

$\Delta_{rel} < 0$

$$\langle \vec{v}\,(0) \cdot \vec{v}\,(t) \rangle = \frac{1}{2} \left(\frac{k_B\,T}{m_0} \right) \left(\frac{1}{\gamma^2\,\rho} \right) \left(\frac{1}{\alpha_{I_{rel}}} \right) \times$$

$$\times \left[(1 + \alpha_{I_{rel}}) \, \exp \left(- \frac{(1 + \alpha_{i\,I})}{2\,\rho} \, \frac{t}{\tau} \right) - \right.$$

$$\left. (1 - \alpha_{I_{rel}}) \, \exp \left(- \frac{(1 - \alpha_{i\,I})}{2\,\rho} \, \frac{t}{\tau} \right) \right] \tag{19}$$

with$\alpha_{I_{rel}} = \sqrt{1 - \frac{4\,\rho\,\omega_0^2\,\tau^2}{\gamma}}$.

The case $\Delta_{rel} = 0$ reduces to relativistic Drude model.

Examples of application

As example of application it has considered the motion of electrons, at different velocities, in a nanostructure of ZnO [27-29]. We underline that the model holds for charged particles in general, not necessarily electrons. Changing the nanomaterial, it will consider the right effective mass and relaxation time. Data to be implemented in Eqs (18,19) are resumed in Table 1.

$v\,(cm/s)$	v/c	$1/\rho$	$\frac{1+\alpha_{I_{Rel}}}{2\,\rho}$ (a)	$\frac{1+\alpha_{I_{Rel}}}{2\,\rho}$ (b)	$\frac{\alpha_{R_{Rel}}}{2\,\rho}$ (20)	$\frac{\alpha_{R_{Rel}}}{2\,\rho}$ (d)
10^7	$0.334{\cdot}10^{-3}$	0.999	0.749	0.549	2.497	9.99
10^8	$0.334{\cdot}10^{-2}$	0.999	0.749	0.549	2.497	9.99
10^9	$0.334{\cdot}10^{-1}$	0.998	0.748	0.549	2.495	9.98
10^{10}	0.334	0.888	0.666	0.488	2.22	8.88
$2{\cdot}10^{10}$	0.834	0.304	0.228	0.167	0.76	3.04

Table 1. Data related to the variation of the carrier velocity v inside the nanostructure.
(a): $\alpha_{I_{Rel}} = 0.5$; (b): $\alpha_{I_{Rel}} = 0.1$; (20): $\alpha_{R_{rel}} = 5$; (d): $\alpha_{R_{rel}} = 20$.

Fig. 1 represents the evolution of $\langle \vec{v}\,(0) \cdot \vec{v}\,(t) \rangle$ in time for a fixed value of $\alpha_{R_{rel}}$ in relation to three different velocities of electrons, with $\tau = 0.84 \cdot 10^{-13}\,s$ and $T = 300\,K$ [27-30]. The classical "Drude" velocity $v = 10^7\,cm/s$ implies a negligible variation in mass for the electrons. We note that the increase in velocity tends to raise the wavelength of the damped oscillation, reducing its amplitude.

In Fig. 2 the same situation is presented, but with a different value of the parameter $\alpha_{R_{rel}}$. The initial more marked compression of the curve (blue solid line) obeys to the same variation indicated in the previous case.

In Figs 3 and 4 the parameter $\alpha_{I_{rel}}$ has been considered. We note as the typical Smith behaviour of $\langle \vec{v}\,(0) \cdot \vec{v}\,(t) \rangle$ tends to become negative in longer times with respect to the classical case and the curves approach the x-axis when the velocity of the carrier increases.

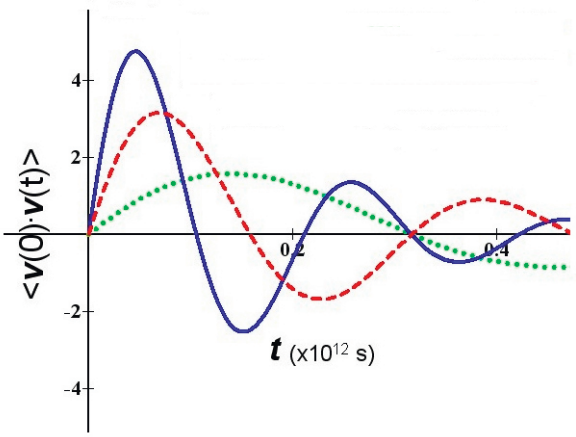

Figure 1: $\langle \vec{v}(0) \cdot \vec{v}(t) \rangle$ vs t for fixed value $\alpha_{R_{rel}} = 5$; the considered velocity of the carrier is $v = 10^7\, cm/s$ (blue solid line), $v = 10^{10}\, cm/s$ (red dashed line) and $v = 2 \cdot 10^{10}\, cm/s$ (green dots line)

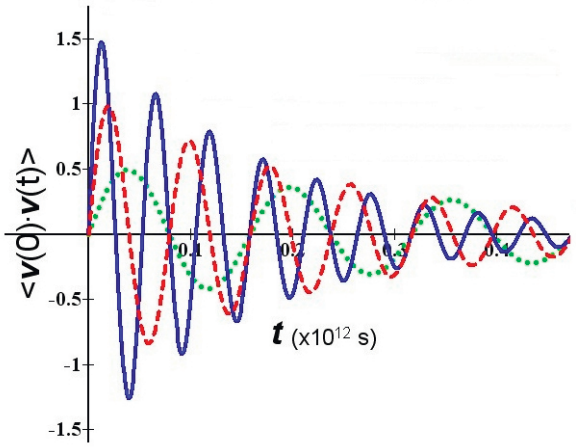

Figure 2: $\langle \vec{v}(0) \cdot \vec{v}(t) \rangle$ vs t for fixed value $\alpha_{R_{rel}} = 20$; the considered velocity of the carrier is $v = 10^7\, cm/s$ (blue solid line), $v = 10^{10}\, cm/s$ (red dashed line) and $v = 2 \cdot 10^{10}\, cm/s$ (green dots line)

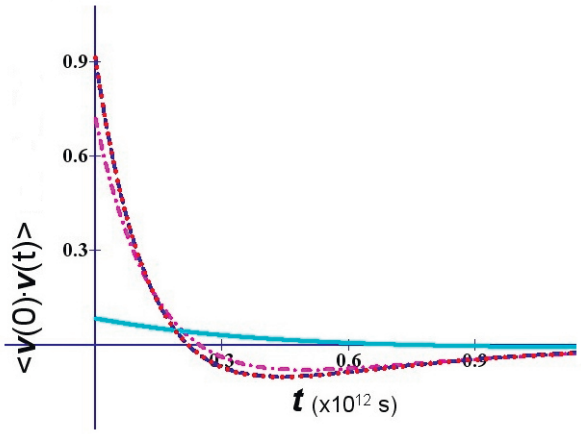

Figure 3: $\langle \vec{v}(0) \cdot \vec{v}(t) \rangle$ vs t for fixed value $\alpha_{I_{rel}} = 0.5$; the considered velocity of the carrier is $v = 10^7\, cm/s$ (blue dashed line) (superposed to the red dots line, representing the classical case), $v = 10^{10}\, cm/s$ (violet dot-dashed line) and $v = 2.5 \cdot 10^{10}\, cm/s$ (clear blue solid line)

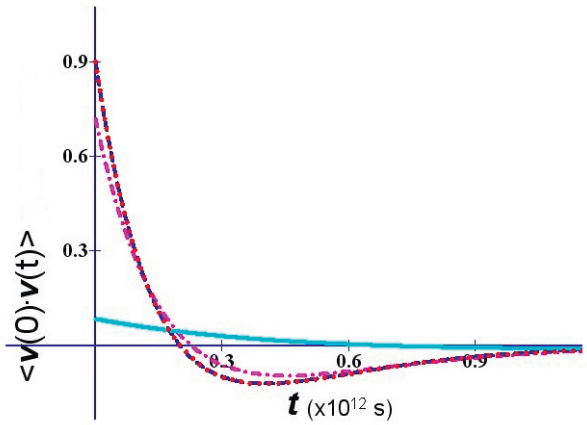

Figure 4: $\langle \vec{v}(0) \cdot \vec{v}(t) \rangle$ vs t for fixed value $\alpha_{I_{rel}} = 0.1$; the considered velocity of the carrier is $v = 10^7\, cm/s$ (blue dashed line) (superposed to the red dots line, representing the classical case), $v = 10^{10}\, cm/s$ (violet dot-dashed line) and $v = 2.5 \cdot 10^{10}\, cm/s$ (clear blue solid line)

Conclusions

In this work it has been initially considered an overview related to the theoretical formulations regulating the dynamics at nanoscale.

Starting by the Drude model, we arrived to an interesting Drude-Lorentz-like model, appeared both in classical and in quantum form, tested in the last years with good accordance with experimental existing data [1,2,14–22,31]; new results regarding the relativistic velocities correlation function have been also presented. The core of the model regards the possibility to obtain the analytical formulation of the most important quantities concerning the dynamics of a system, through the complete Fourier transform of the real part of the frequency-dependent complex conductivity $\sigma(\omega)$, extending the integration over time on the entire time axis (-∞, +∞). This extension is mathematically very elegant, because of the analytical approach, and gives interesting and new informations about the dynamics of systems through the elaboration of experimental data.

At applied level, we suggest the possibility to a fast/ultrafast injection of carriers in a nanostructure for possible practical needs of raising the wavelength of the damped oscillation for the velocities correlation function and reducing its amplitude.

The complete development of the relativistic calculation of R^2 and D will provide interesting peculiarities and new results, like the found time oscillations in velocity at beginning of process for the quantum non-relativistic version [1,14,16,31], which could be appropriately tested through experimental time-resolved techniques, like TRTS [32–34].

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Classical model of quantumness of bosonic strings

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Abstract: *In this paper we will show how to model quantum mechanical probability amplitude of behavior of strings classically*

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Introduction

One of the problems of interpretation of quantum mechanics is how can a particle go through both slits "at the same time". In the single particle case, this question has an answer: *classical* electromagnetic field corresponds to a photon being at different places at the same time, yet it makes perfect sense. Conversely, an electron being in different places at the same time corresponds to field ψ that can also be understood as "classical". This analogy breaks down, however, when we have multiparticle system. How can we have different *configurations of* particles at the same time is far more difficult to answer. After all, this would correspond to a wave function over $\mathbb{R}^{3n+1}$ while classical electromagnetic field is a function over $\mathbb{R}^4$.

However, what "saves" us in case of string theory is that, due to highly non-trivial topology (handles, and so forth) string worldsheet forms a network. Now, we can divide a network into sub-networks and assume that their number is so large that they coarse grain the set of all possible sub-networks one can imagine. In this case, one can produce a network by simply making a set of decisions as to which sub-network to include and which to excluse. There is one constraint: each sub-network can be either fully included or fully excluded. Thus, we include the entire sub-network if and only if we include a given atom of said sub-network. *But* the probability amplitude of including of said atom *does* have classical analogue: its a *complex valued* temperature of that atom! Thus, we have a classical definition of probability amplitude of including of *entire* sub-network. Finally, in order for said definition to be consistent, we have to assume that the temperatures of any two atoms of a given sub-network are the same.

One example of this is the city being the network of streets. An atom is a piece of sand in that city, and the temperature of an arbitrary piece of sand in that city is the probaiblity of that network of streets being selected. Since we know the city limits, we know that piece of sand in cremlin selects the entire city of Moscow, rather than just Cremlin. Furthermore, a piece of sand at the North of Moscow and piece of sand in the South of Moscow both select exact same network: an entire city of Moscow. Thus, all pieces of sand throughout Moscow have exact same temperature. The explanation for this is that city of Moscow is being surrounded by insulator; yet, the heat is free to diffuse throughout Moscow.

Now we ran into problem: heat diffusion happens only across the space coordinates. In other words, in case of Moscow (where both North/South and East/West are "spacelike") we are fine; but in case of string, where σ is space and τ is time, we run into difficulty. We reconcile it by saying that the "conventional" idea that τ is time is "wrong": τ is in fact space! Now, in order to still have causality, we have a *new* "time" coordinate ξ, which is altogether absent in "conventional" framework (after all, in case of Moscow we need time t to describe heat diffusion that would lead to desired equilibrium situation). Thus, we have two "space" coordinates, (σ, τ), and one "time" coordinate ξ. After that, we impose a set of initial conditions that would result in "correlation" between ξ and τ, thus explaining the "illusion" of timelike nature of the latter. Without said initial conditions, ξ and τ are orthogonal, which is why τ is "space".

Let us illustrate the idea by the following example. Suppose we have four sub-networks, A, B, C and D. Now, suppose sub-network A connects to B and C, and both of those networks connect to D. Finally, suppose that A corresponds to $\tau_1 < \tau < \tau_2$, B and C both correspond to $\tau_2 < \tau < \tau_3$ and D corresponds to $\tau_3 < \tau < \tau_4$. Finally, suppose we put heat insulators across the lines $\tau = \tau_1$, $\tau = \tau_2$, $\tau = \tau_3$ and $\tau = \tau_4$. In other words

each branch of a network that "happens" to cross one of these lines will be divided by an insulator at that line. Now, suppose that at a time $\xi = \xi_1$ we put a heat sourse at $\tau = \tau_1^+$. In other words the heat source is displaced infinitesimally from $\tau = \tau_1$ insulator towards $+\tau$ direction; as a result, the insulator will prevent heat from diffusion into $\tau < \tau_1$ terriotory, but it would freely diffuse into $\tau > \tau_1$ region. Now, at time $\xi = \xi_2$, the heat evens out througout region A, and the common value of that heat is $\psi(A)$. We then introduce new heat sources, at $\tau = \tau_2^+$. The heat sources at $A \cap B$ have strength $\psi(A)e^{iS(B)}$ while the ones at $A \cap C$ have strength $\psi_A e^{iS(C)}$. As a result, at time $\xi = \xi_3$, the common temperature throughout regions B and C will be

$$\psi(B, \xi_3) = k_{AB}\psi(A, \xi_2)e^{iS(B)} \ , \ \psi(C, \xi_3) = k_{AC}\psi(A, \xi_2)e^{iS(C)} \tag{1}$$

And, finally, at $\xi = \xi_3$, we introduce heat sources at $\tau = \tau_3^+$: at the $B \cap D$ and $C \cap D$ parts of the region the strength of the sources will be $\psi(B)e^{iS(D)}$ and $\psi(C)e^{iS(D)}$, respectively. Then, at $\xi = \xi_4$, the equilibrium will be reached throughout D with the common value of ψ being

$$\psi(D, \xi_4) = (k_{BD}\psi(B) + k_{CD}\psi(C))e^{iS(D)} = \\ \psi(A, \xi_2)(k_{AB}k_{BD}e^{i(S(B)+S(D))} + k_{AC}k_{CD}e^{i(S(C)+S(D))}) \tag{2}$$

Finally, since $\psi(A)$ has reached equilibrium at $\xi = \xi_2$, we know that

$$\psi(A, \xi_4) = \psi(A, \xi_2) \tag{3}$$

Thus, we can re-write Eq 2 in the form of

$$\psi(D, \xi_4) = \psi(A, \xi_4)(k_{AB}k_{BD}e^{i(S(B)+S(D))} + k_{AC}k_{CD}e^{i(S(C)+S(D))}) \tag{4}$$

Now, comparing Eq 2 and Eq 4 gives us interesting inside. On the one hand, Eq 2 shows the "timelike" nature of τ in a sense that $\tau \in (\tau_3, \tau_4)$ is "coupled" to $\xi = \xi_4$ whereas $\tau \in (\tau_1, \tau_2)$ is "coupled" to $\xi = \xi_2$. On the other hand, Eq 4 shows the "spacelike" nature of τ: both ranges of τ correspond to $\xi = \xi_4$. Yet, it is also clear why it "falsely appears" that τ is time: the spacelike (!!!) picture at $\xi = \xi_4$ happens to resemple the space-time (!!!) path integral. The "source" of timelike appearance is initial conditions: in particular, the initial conditions at $\tau = \tau_1$, $\tau = \tau_2$ and $\tau = \tau_3$ were specified at $\xi = \xi_1$, $\xi = \xi_2$ and $\xi = \xi_3$, respectively. This, together with the fact that the initial conditions were set at τ_k^+ instead of τ_k^- is what makes τ to look like "future" direction.

One thing that might strike the reader is that diffusion equaitons would require $(+,+)$ metric, in contradiction to $(+,-)$ worldsheet metric. The answer to this question is that the "history" as we know it is spacelike (σ, τ) surface taken at $\xi \to \infty$; thus, the diffusion process and, therefore, $(+,+)$ metric, are not relevent any more since equilibrium has been reached. On the other hand, $(+,-)$ metric figures in the equations for $S(A)$, $S(B)$, $S(C)$ and $S(D)$ above and, therefore, continues to be relevent, leading to the appearance of a single $(+,-)$ metric. On the other hand, when ξ is finite, both diffusion as well as S are relevent, thus both metric take place. This implies that worldsheet symmetry is broken. At the same time, the symmetry of X^μ, given by $\eta_{\mu\nu}$ continues to hold. Thus, we have three metrics:

$$h_{\tau\tau} = h^{\tau\tau} = 1 \ , \ h_{\sigma\sigma} = h^{\sigma\sigma} = -1 \ , \ e_{\tau\tau} = e^{\tau\tau} = e_{\sigma\sigma} = e^{\sigma\sigma} = 1 \tag{5}$$

$$h_{\sigma\tau} = h^{\sigma\tau} = h_{\tau\sigma} = e^{\tau\sigma} = e_{\sigma\tau} = e^{\sigma\tau} = e_{\tau\sigma} = e^{\tau\sigma} = 0 \tag{6}$$

$$\eta_{00} = \eta^{00} = 1 \ , \ \eta^{11} = \eta_{11} = \eta^{22} = \eta_{22} = \eta^{33} = \eta_{33} = -1 \tag{7}$$

$$\eta_{j0} = \eta^{j0} = \eta_{0j} = \eta^{0j} = \eta_{ij} = \eta^{ij} = 0 \ , \ i \neq 0 \ , \ j \neq 0 \ , \ i \neq j \tag{8}$$

Heat conduction process

Let us now describe diffusion process more explicitly. We will introduce two kinds of "heat": S and ψ. Our goal is for these two quantities to obey, in $\xi \to \infty$, the following:

$$\lim_{\xi\to\infty} \psi(A) \approx \lim_{\xi\to\infty} \left(e^{iS(A)} \sum_{B \prec^* A} \psi(B) \right) \tag{9}$$

where

$$\lim_{\xi\to\infty} S(\sigma,\tau;\xi) = \lim_{\xi\to\infty} \left(\frac{1}{2\pi\alpha'} \int_{V(\sigma,\tau)} d\sigma' d\tau' g^{1/2} g^{\alpha\beta} \partial_a X^\mu \partial_b X_\mu + \right.$$

$$\left. + \frac{1}{4\pi} \int_{V(\sigma,\tau)} d\sigma' d\tau' g^{1/2} R + \frac{1}{2\pi} \int_{\partial V(\sigma,\tau)} ds\ k \right) \tag{10}$$

where $V(\sigma,\tau)$ is a sub-string that contains a given point (σ,τ). This is what allows the left hand side to be a function of a *point* (σ,τ), thus have a *classical* meaning and, *at the same time*, represent the probability amplitude of a sub-network (or sub-string) since $V(\sigma,\tau)$ is sub-network, which is perfectly consistent with (σ,τ) being a point.

The Eq 10 has simply been copied from p.82 of Polchinski, [1], with two changes. First, we replace the entire string worldsheet with a relevent sub-string. Secondly, we claim that the approximation holds *only* as an equilibrium, in a limit of time ξ being infinite, while in conventional string theory time ξ is absent altogether and the above holds without the need of "equilibrium". Now, the diffusion process that would create the above situation has to have the following components:

a) The parameter S has to diffuse across the sub-string

b) The "sources" of S have to include $\mathcal{L}$, R, and k

c) The "insulator" $A \cap B$ should have a "source" of $\psi(B)$ but *not* of $\psi(A)$. This can be accomplished by placing the source infinitesimally in the B-direction from the insulator; in other words, it is placed at $(A \cap B)^+$ rather than $(A \cap B)^-$.

d) The strength of the "source" mentioned in part d has to be proportional to $e^{iS_{+\tau}} \psi_{-\tau}$

The parts a-b are meant to produce Eq 10 while Eq c-d are meant to produce Eq 9. For simplicity, let us rewrite Eq 10 as

$$\lim_{\xi\to\infty} S = \lim_{\xi\to\infty} \int_V d\sigma d\tau\ \mathcal{L}(\sigma,\tau) \tag{11}$$

where we use δ-functions in order to absorb curvature term into $\mathcal{L}$:

$$\mathcal{L}(\sigma,\tau) = \frac{1}{2\pi\alpha'} g^{1/2} g^{\alpha\beta} \partial_a X^\mu \partial_b X_\mu + \frac{1}{4\pi} g^{1/2} R + \frac{1}{2\pi} \int_{\partial V(\sigma,\tau)} ds\ k(s)\ \delta^2_{\epsilon_1}(\sigma - \sigma(s)) \tag{12}$$

where

$$\delta_\epsilon(x) = \sqrt{2\pi\epsilon} e^{-\frac{x^2}{2\epsilon}}\ ,\ \delta^2_\epsilon(x,y) = \delta_\epsilon(x)\delta_\epsilon(y) \tag{13}$$

Parts a-b can be enforced through diffusion process in "space" coordinates (σ,τ) that is taking place in "time" ξ,

$$\frac{\partial S}{\partial \xi} = e^{\alpha\beta} \partial_\alpha (f \partial_\beta S)\ ,\ f = \exp\left(- \int_{\partial V(\sigma,\tau)} ds\ k(s)\ \delta^2_{\epsilon_2}(\sigma - \sigma(s)) \right) \tag{14}$$

with the initial conditions

$$S(\sigma,\tau;\xi = \xi_1) = \int_{\partial V(\sigma,\tau)} ds\ k(s)\ \delta^2_{\epsilon_3}(\sigma - \sigma(s)) \tag{15}$$

In the Eq 14 the function f corresponds to the variation of conductivity; the $f \to 0$ limit corresponds to insulator. Now, due to the δ-function in Eq 14 we see that material has constant conductivity away from the boundary, yet it becomes insulator in the vicinity of the boundary. The fact that $f \to 0$ at the boundary happens at exponential rate implies that, provided that S is well behaved, all derivatives of f are zero at the boundary; in other words,

$$\left[\forall(i,j)\partial_\sigma^i\partial_\tau^j S\Big|_{\partial V} \neq \infty\right] \Longrightarrow \frac{\partial S}{\partial \xi}\Big|_{\partial V} = 0 \tag{16}$$

This feauture leads to boundary conditions; namely, the behavior of S at the boundary at arbitrary ξ needs to be the same as it used to be at $\xi = \xi_0$. Another consequence of exponential behavior of f at the boundary is that, by Gauss' theorem,

$$f\partial_\beta S\Big|_{\partial V} = 0 \Longrightarrow \int e^{\alpha\beta}\partial_\alpha(f\partial_\beta S) = 0 \Longrightarrow \frac{\partial}{\partial\xi}\int S(\sigma,\tau;\xi)d\sigma d\tau = 0 \tag{17}$$

In light of Eq 15 this becomes

$$\forall\xi\left(\int S(\sigma,\tau)d\sigma d\tau \approx \int_{\partial V(\sigma,\tau)} ds\ k(s)\right) \tag{18}$$

Now, our physical knowledge of diffusion tells us that, at $\xi \to \infty$, the value of S approaches constant. This means that

$$\lim_{\xi\to\infty} S(\sigma,\tau;\xi) \approx \frac{1}{Vol(V(\sigma,\tau))}\int_{\partial V(\sigma,\tau)} ds\ k(x) \tag{19}$$

In particular, we can find $\xi_V \gg \xi_0$ such that

$$S(\sigma,\tau;\xi_2) \approx \frac{1}{Vol(V(\sigma,\tau))}\int_{\partial V(\sigma,\tau)} ds\ k(x) \tag{20}$$

where the value of ξ_V will differ for each region V; namely, if B has larger τ than A, then $\xi_B > \xi_A$; yet, the values of ξ_0 are the same for all regions.

The above results in parts a-b being true. Now let us produce parts c-d. In order to do that, we postulate similar diffusion equation for ψ,

$$\frac{\partial\psi}{\partial\xi} = e^{\alpha\beta}\partial_\alpha(f\partial_\beta\psi) \tag{21}$$

but this time we will impose initial condition at $\xi = \xi_V$ *as opposed to* $\xi = \xi_0$. Thus, we will be able to assume that S is constant throughout the substring (see Eq 20) and focus entirely on behavior of ψ. Thus, we will set our boundary conditions to be

$$\psi(\sigma,\tau;\xi=\xi_2) = \sum_{W_k\cap V\neq\emptyset; W_k\prec V}\int_{\partial V\cap W} ds\ \psi^-(s)e^{iS^+(s)}\ \delta^2_{\epsilon_3}(\sigma-\sigma(s)) \tag{22}$$

where $A \prec B$ means that one can draw a curve starting at an element of A and ending in the element of B, which moves in the direction of increase of τ; and we further assume that whenever $A \prec B$ holds, $B \prec A$ does not. Furthermore, by $\psi^-(s)$ we mean the following: since $W_k \cap V$ is an insulator, the behavior of ψ is discontinuous across that line. Now, the value of $\psi^-(s)$ is measured infinitesimally towards the W_k side, while the value of $\psi^+(s)$ is measured infinitesimally to the V side; both statements are true regardless of which direction the point s, itself, is shifted. In our case, s is shifted infinitesimally to V side; thus $\psi^-(s)$ implies "infinitesimal nonlocality". Now, suppose the value of ξ_V is selected in such a way that, at the time $\xi = \xi_V$, the field ψ has already reached its respective "constant" values at every single region W_k (thus, $\xi_{W_k} < \xi_V$); we will denote the latter "constant" by $\psi(W_k)$. It is then easy to see that

$$\lim_{\xi\to\infty}\psi(V;\xi) = \sum_{W_k\cap V\neq\emptyset; W_k\prec V}\lambda(W_k\cap V)\psi(W_k,\xi_2)e^{iS(V)} \tag{23}$$

where $\lambda(W_k \cap V)$ denotes the length of $W_k \cap V$. It is true that the limit on left hand side is achieved only for $\xi \gg \xi_V$. At the same time, however, the right hand side has already achieved said limit at $\xi = \xi_V$. Thus, we can replace $\xi = \xi_V$ with $\xi \to \infty$ on the right hand side, to produce

$$\lim_{\xi\to\infty} \psi(V;\xi) = \lim_{\xi\to\infty} \left(\sum_{W_k\cap V\neq\emptyset; W_k \prec V} \lambda(W_k \cap V)\psi(W_k,\xi)e^{iS(V)} \right) \tag{24}$$

If we now assume that there are constants λ and V such that

$$\forall A \prec B(\lambda(A \cap B) = \lambda \ , \ V(A) = V(B) = V) \tag{25}$$

Then Eq 20 together with Eq 24 would produce the desired evolution of ψ *in* τ: after all, V is displaced *in* $+\tau$ *direction* from W_k. Yet, such result is only an emergent $\xi \to \infty$ limit of Eq 14 and 21 under the initial conditions 15 and . The "timelike" nature of τ is enforced by the fact that $\xi_V \gg \xi_{W_k}$ for all k. This is seen from the fact that we have assumed that ψ has already reached equilibrium throught W_k at the time $\xi = \xi_2(V)$, as evident from our use of notation $\psi(W_k)$ in Eq 23. This, in combination with the fact that V is shifted in $+\tau$ direction relative to W_k produces the correlation between τ and ξ *in initial conditions* which is what makes τ appear "timelike" in the outcome. In addition, the "timelike" nature of τ is also enforced in the use of ψ^-, instead of ψ^+, in Eq .

Conclusions

To sum it up, we have produced a coarse graining of quantum mechanical process on the string. Coarse graining consists of a set of choices of sub-networks within an afore-existing "big" network and afore-existing criteria that each sub-network can either be fully included or fully excluded. The union of sub-networks that are fully included would represent "history". The fact that "big" network has been fixed is what allows us to avoid "true" quantization in favor of "emergent" one; the latter can be assigned realist meaning while the former can not. Furthermore we have re-interpretted the notion of spacetime. We view (σ, τ) coordinates as space-only. The "timelike" nature of τ is due to a "pattern" displayed in otherwise "space-alone" plane. That pattern had emerge over the evolution of such plane in ξ at a limit $\xi \to \infty$. The overall achievement of this work is that the probability amplitudes on the trajectory space are being interpretted in terms of "heat" in the *ordinary* space (namely, the surface of the *fixed* world-sheet).

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Role of spinor field in isotropization of initially anisotropic spacetime

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Abstract: *Within the scope of an anisotropic Bianchi type-I Universe we study the role of spinor field in the evolution of the Universe. It is found that due to the nontrivial non-diagonal components of the energy momentum tensor of the spinor filed there exist two possibilities. In one scenario the initially anisotropic Universe evolves into an isotropic one asymptotically, but in this case the spinor field itself undergoes some severe restrictions. In the second scenario the isotropization takes places almost at the beginning of the process.*

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Introduction

Recent observation shows that the Universe is expanding with acceleration [1,2]. Though there exist a number of models able to explain this late time acceleration, a number of authors introduced the spinor field into the system in order to study this phenomenon [3–16]. It was found that the spinor field for a suitable choice of nonlinearity can

(i) give rise to singularity free Universe;
(ii) accelerate the isotropization process;
(iii) generate accelerating mode of expansion of the Universe.

It was also found that the nonlinear spinor field can simulate different kinds of matter field ranging from ekpyrotic matter to phantom matter as well as Chaplygin gas [17–21].

Basic equations

The spinor field Lagrangian we take in the form [22]

$$L_{\rm sp} = \frac{\imath}{2}\left[\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi\right] - m_{\rm sp}\bar{\psi}\psi - F, \tag{1}$$

where the nonlinear term F describes the self-interaction of a spinor field and can be presented as some arbitrary functions of invariants generated from the real bilinear forms of a spinor field. In our case we consider $F = F(K), \quad K = \{I, J, I \pm J\}$ where $I = S^2 = (\bar{\psi}\psi)^2, \quad J = P^2 = (\imath\bar{\psi}\gamma^5\psi)^2$. In (1) ∇_{μ} is the covariant derivative of spinor field:

$$\nabla_{\mu}\psi = \partial_{\mu}\psi - \Gamma_{\mu}\psi, \quad \nabla_{\mu}\bar{\psi} = \partial_{\mu}\bar{\psi} + \bar{\psi}\Gamma_{\mu}, \tag{2}$$

where Γ_{μ} is the spinor affine connection.

Variation with respect to $\bar{\psi}(\psi)$ gives spinor field equations:

$$\imath\gamma^{\mu}\nabla_{\mu}\psi - m_{\rm sp}\psi - 2F_K(SK_I + \imath PK_J\gamma^5)\psi = 0, \tag{3a}$$
$$\imath\nabla_{\mu}\bar{\psi}\gamma^{\mu} + m_{\rm sp}\bar{\psi} + 2F_K\bar{\psi}(SK_I + \imath PK_J\gamma^5) = 0. \tag{3b}$$

where we denote $F_K = dF/dK$, $K_I = dK/dI$ and $K_J = dK/dJ$.

It can be shown that on account of spinor field equations (3) the Lagrangian (1) takes the following form

$$\begin{aligned} L_{\rm sp} &= \frac{\imath}{2}\big[\bar\psi\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma^\mu\psi\big] - m_{\rm sp}\bar\psi\psi - F(K) \\ &= \frac{\imath}{2}\bar\psi[\gamma^\mu\nabla_\mu\psi - m_{\rm sp}\psi] - \frac{\imath}{2}[\nabla_\mu\bar\psi\gamma^\mu + m_{\rm sp}\bar\psi]\psi - F(K), \\ &= 2F_K(IK_I + JK_J) - F(K) = 2KF_K - F(K). \end{aligned} \tag{4}$$

The energy-momentum tensor of the spinor field has the form

$$T^\rho_\mu = \frac{\imath}{4}g^{\rho\nu}\big(\bar\psi\gamma_\mu\nabla_\nu\psi + \bar\psi\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma_\nu\psi - \nabla_\nu\bar\psi\gamma_\mu\psi\big) - \delta^\rho_\mu L_{\rm sp} \tag{5}$$

which on account of (2) can be written as

$$\begin{aligned} T_\mu{}^\rho &= \frac{\imath}{4}g^{\rho\nu}\big(\bar\psi\gamma_\mu\partial_\nu\psi + \bar\psi\gamma_\nu\partial_\mu\psi - \partial_\mu\bar\psi\gamma_\nu\psi - \partial_\nu\bar\psi\gamma_\mu\psi\big) \\ &- \frac{\imath}{4}g^{\rho\nu}\bar\psi\big(\gamma_\mu\Gamma_\nu + \Gamma_\nu\gamma_\mu + \gamma_\nu\Gamma_\mu + \Gamma_\mu\gamma_\nu\big)\psi \\ &- \delta^\rho_\mu\big(2KF_K - F(K)\big). \end{aligned} \tag{6}$$

From (6) it becomes clear that if the spin connections Γ_μ's for a given metric are different, the spinor field in that case possesses non-diagonal components of EMT.

The spinor affine connection matrices $\Gamma_\mu(x)$ are uniquely determined up to an additive multiple of the unit matrix by the equation

$$\frac{\partial\gamma_\nu}{\partial x^\mu} - \Gamma^\rho_{\nu\mu}\gamma_\rho - \Gamma_\mu\gamma_\nu + \gamma_\nu\Gamma_\mu = 0, \tag{7}$$

with the solution

$$\Gamma_\mu = \frac{1}{4}\bar\gamma_a\gamma^\nu\partial_\mu e^{(a)}_\nu - \frac{1}{4}\gamma_\rho\gamma^\nu\Gamma^\rho_{\mu\nu}. \tag{8}$$

For the Bianchi type -I cosmological model given by

$$ds^2 = dt^2 - a_1^2\,dx^2 - a_2^2\,dy^2 - a_3^2\,dz^2, \tag{9}$$

with a_i being the functions of time only, the $\Gamma_\mu(x)$ have the form

$$\Gamma_0 = 0, \quad \Gamma_1 = \frac{\dot a_1}{2}\bar\gamma^1\bar\gamma^0, \quad \Gamma_2 = \frac{\dot a_2}{2}\bar\gamma^2\bar\gamma^0, \quad \Gamma_3 = \frac{\dot a_3}{2}\bar\gamma^3\bar\gamma^0. \tag{10}$$

Energy momentum tensor

The nonzero components of the energy momentum tensor in this case read

$$T^0_0 = m_{\rm sp}S + F(K), \tag{11a}$$

$$T^1_1 = T^2_2 = T^3_3 = F(K) - 2KF_K, \tag{11b}$$

$$T^1_2 = \frac{\imath}{4}\frac{a_2}{a_1}\left(\frac{\dot a_1}{a_1} - \frac{\dot a_2}{a_2}\right)\bar\psi\bar\gamma^1\bar\gamma^2\bar\gamma^0\psi, \tag{11c}$$

$$T^1_3 = \frac{\imath}{4}\frac{a_3}{a_1}\left(\frac{\dot a_3}{a_3} - \frac{\dot a_1}{a_1}\right)\bar\psi\bar\gamma^3\bar\gamma^1\bar\gamma^0\psi, \tag{11d}$$

$$T^2_3 = \frac{\imath}{4}\frac{a_3}{a_2}\left(\frac{\dot a_2}{a_2} - \frac{\dot a_3}{a_3}\right)\bar\psi\bar\gamma^2\bar\gamma^3\bar\gamma^0\psi. \tag{11e}$$

As one sees, in case of a FRW cosmological model where $a_1 = a_2 = a_3$ the non-diagonal components of the energy momentum tensor vanish.

Einstein Equation

Taking into account that the metric (9) has only diagonal non-zero components of Einstein tensor, the corresponding system of Einstein equations has the form

$$\begin{aligned}
\frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} &= \kappa T_1^1, && \text{(12a)}\\
\frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_1}{a_1} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1} &= \kappa T_2^2, && \text{(12b)}\\
\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} &= \kappa T_3^3, && \text{(12c)}\\
\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1} &= \kappa T_0^0, && \text{(12d)}
\end{aligned}$$

which should be supplemented by additional conditions

$$\begin{aligned}
\frac{\imath}{4}\frac{a_2}{a_1}\left(\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2}\right)\bar{\psi}\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^0\psi &= 0, && \text{(13a)}\\
\frac{\imath}{4}\frac{a_3}{a_1}\left(\frac{\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1}\right)\bar{\psi}\bar{\gamma}^3\bar{\gamma}^1\bar{\gamma}^0\psi &= 0, && \text{(13b)}\\
\frac{\imath}{4}\frac{a_3}{a_2}\left(\frac{\dot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3}\right)\bar{\psi}\bar{\gamma}^2\bar{\gamma}^3\bar{\gamma}^0\psi &= 0. && \text{(13c)}
\end{aligned}$$

The conditions (13) open two possibilities:

$$\bar{\psi}\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^0\psi = \bar{\psi}\bar{\gamma}^3\bar{\gamma}^1\bar{\gamma}^0\psi = \bar{\psi}\bar{\gamma}^2\bar{\gamma}^3\bar{\gamma}^0\psi = 0, \tag{14}$$

which imposes conditions of the components of the spinor field leaving the metric unchanged, or

$$\left(\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2}\right) = \left(\frac{\dot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3}\right) = \left(\frac{\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1}\right) = 0, \tag{15}$$

that makes the space-time isotropic from the very beginning.

Two possibilities

Let us consider the first case when spinor field obeys (8). On view of the fact that $T_1^1 = T_2^2 = T_3^3$ from (12) one dully finds

$$a_i = D_i V^{1/3} \exp\left(X_i \int \frac{dt}{V}\right), \quad \prod_{i=1}^{3} D_i = 1, \quad \sum_{i=1}^{3} X_i = 0, \tag{16}$$

with D_i and X_i being the integration constants. Thus we see that the metric functions can be expressed in terms of V.

Summation of (12a), (12b), (12c) and 3 times (12d) leads to the equation for V

$$\ddot{V} = 3\kappa(F(K) - KF_K)V, \tag{17}$$

where for simplicity we set $m_{\rm sp} = 0$. Let us note that, in the unified nonlinear spinor theory of Heisenberg, the massive term remains absent, and according to Heisenberg, the particle mass should be obtained as a result of quantization of spinor prematter [23,24]. In the nonlinear generalization of classical field equations, the massive term does not possess the significance that it possesses in the linear one, as it by no means defines total energy (or mass) of the nonlinear field system. Thus without losing the generality we can consider the massless spinor field putting $m = 0$. From the spinor field equations it can be shown that $K = K_0^2/V^2, \quad K_0 = \text{const}$.

The Eq. (17) can be solved in quadrature. Let us go back to the spinor field equations. For simplicity we consider the case when $K = I$. In this case (3a) can be written as

$$\dot{\phi}_1 + \imath\mathcal{D}\phi_1 = 0, \tag{18a}$$
$$\dot{\phi}_2 + \imath\mathcal{D}\phi_2 = 0, \tag{18b}$$
$$\dot{\phi}_3 - \imath\mathcal{D}\phi_3 = 0, \tag{18c}$$
$$\dot{\phi}_4 - \imath\mathcal{D}\phi_4 = 0, \tag{18d}$$

where $\phi_i = \sqrt{V}\psi_i, \quad \psi = \text{col}(\psi_1, \psi_2, \psi_3, \psi_4), \quad \mathcal{D} = 2SF_K$.

As one sees, the foregoing system of equations is easily solvable. In this case for the components of spinor field we find

$$\psi_1(t) = (C_1/\sqrt{V})\exp\big(-i\int \mathcal{D}dt\big), \tag{19a}$$
$$\psi_2(t) = (C_2/\sqrt{V})\exp\big(-i\int \mathcal{D}dt\big), \tag{19b}$$
$$\psi_3(t) = (C_3/\sqrt{V})\exp\big(i\int \mathcal{D}dt\big), \tag{19c}$$
$$\psi_4(t) = (C_4/\sqrt{V})\exp\big(i\int \mathcal{D}dt\big), \tag{19d}$$

with C_1, C_2, C_3, C_4 being the integration constants and related to K_0 as $C_1^*C_1 + C_2^*C_2 - C_3^*C_3 - C_4^*C_4 = K_0$.

Recalling that the spinor field should satisfy the equality (8), after some manipulations we get

$$C_1^*C_2 + C_3^*C_4 = C_2^*C_1 + C_4^*C_3 = 0, \tag{20a}$$
$$C_1^*C_1 - C_3^*C_3 = C_2^*C_2 - C_4^*C_4 = \frac{K_0}{2}. \tag{20b}$$

For the nonlinear term

$$F(K) = \lambda K^{(1+W)/2}, \quad \lambda = \text{const}. \tag{21}$$

which describes a perfect fluid from ekpyrotic matter to phantom we get

$$\int \frac{dV}{\sqrt{3\kappa\lambda V_0^{1+W}V^{1-W} + C_1}} = t + t_0, \tag{22}$$

whereas, for the nonlinear term

$$F = \big(A + \lambda K^{(1+\gamma)/2}\big)^{1/(1+\gamma)}. \tag{23}$$

which describes a generalized Chaplygin gas we get

$$\int \frac{dV}{\sqrt{C_1 + 3\kappa V\big(AV^{1+\gamma} + \lambda V_0^{1+\gamma}\big)^{1/(1+\gamma)}}} = t + t_0. \tag{24}$$

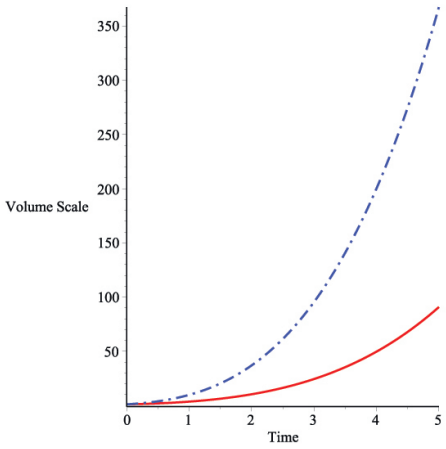

Figure 1: Evolution of the Universe filled with quintessence. The solid (red) line stands for volume scale V, while the dash-dot (blue) line stands for a^3.

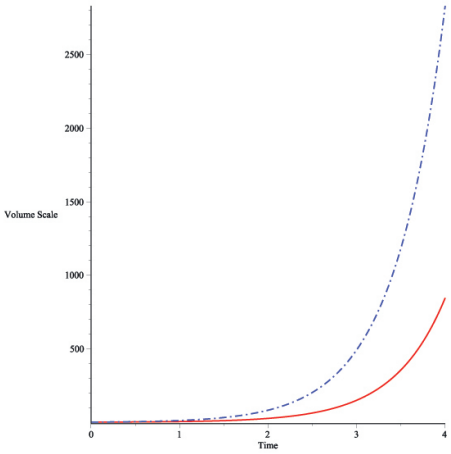

Figure 2: Evolution of the Universe filled with Chaplygin gas. The solid (red) line stands for volume scale V, while the dash-dot (blue) line stands for a^3.

If we explore the second possibility, the non-diagonal components of the EMT imposes restriction on the metric functions give by (15). In that case we have

$$\frac{\dot{a}_1}{a_1} = \frac{\dot{a}_2}{a_2} = \frac{\dot{a}_3}{a_3} \equiv \frac{\dot{a}}{a}. \tag{25}$$

The Einstein system in this case takes the form

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \kappa T_1^1, \tag{26a}$$

$$3\frac{\dot{a}^2}{a^2} = \kappa T_0^0, \tag{26b}$$

which describes a FRW model.

In order to find the solution that satisfies both (26a) and (26b) we rewrite (26a) in view of (26b) in the following form:

$$\ddot{a} = \frac{\kappa}{6}\Big(3T_1^1 - T_0^0\Big)a. \tag{27}$$

As in previous case, for the given nonlinearity the solution to (27) can be found in quadrature. The components of metric functions in this case read

$$a_i = D_i V^{1/3} = D_i a, \quad \prod_{i=1}^{3} D_i = 1, \tag{28}$$

which means it represents a tiny sector of the general solutions (16) which one obtains for the BI model in case of isotropic distribution of matter with trivial non-diagonal components of energy-momentum tensor, e.g., when the Universe is filled with perfect fluid, dark energy etc.

Conclusions

1. Within the scope of Bianchi type-I space time we study the role of spinor field on the evolution of the Universe.
2. It is shown that the spinor field possesses non-zero non-diagonal components of energy-momentum tensor thanks to its specific relation with gravitational field. This fact plays vital role on the evolution of the Universe.
3. There might be two different scenarios: (i) the components of the spinor field are affected leaving the space-time initially anisotropic that evolves into an isotropic one asymptotically; (ii) the space-time becomes isotropic right from the beginning and can be completely described by the Einstein field equations for FRW metric.
4. As numerical analysis shows, in case of early isotropization the Universe expands rather rapidly.
5. There might be another possibility when the non-diagonal components of energy-momentum tensor influence both the spinor field and metric functions simultaneously.

Acknowledgments

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Cosmological dynamics in the model with nonminimal kinetic coupling

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Abstract: *Cosmological dynamics has been considered for the FRW Universe filled with a scalar field φ with the kinetic coupling $\kappa G_{\mu\nu}\varphi^{,\mu}\varphi^{,\nu}$ and the effective potential $V(\varphi) = V_0\varphi^N$. The influence of such potential on the dynamics of the model is found using the methods of the theory of dynamical systems. Two stable asymptotic regimes are obtained: one exists only for $0 < N < 2$ and the other one for $N > 2$. It is also shown that realistic cosmological scenarios are possible in this model.*

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Introduction

Theories with nonminimal kinetic coupling of a scalar field and gravity belong to a class of the most general scalar-tensor theories. In the context of inflationary cosmology these models were proposed by Amendola in 1993 [1].

In the theories of this type the cosmological action contains combinations of the components of the curvature tensor and derivatives of a scalar field: $\kappa_1 R\phi_{,\mu}\phi^{,\mu}$, $\kappa_2 R_{\mu\nu}\phi_{,\mu}\phi^{,\nu}$, $\kappa_3 R\phi\Box\phi$ and others, where κ_i are the nonminimal kinetic coupling constants. The Lagrangian giving the second-order equations of motion was derived by Horndeski in 1974 [2].

One of the most of intriguing feature of the models with the kinetic coupling $\kappa G_{\mu\nu}\phi^{,\mu}\phi^{,\nu}$ ($G_{\mu\nu}$ is the Einstein tensor) found in [3]-[5] is the existence of inflationary behaviour at the early time in the case of zero or constant potential of the scalar field, i. e. solely due to the coupling. This regime exists only for a positive coupling constant κ.

In this paper the model of the Universe with the nonminimal kinetic coupling of the scalar field ϕ with gravity of the form $\kappa G_{\mu\nu}\phi^{,\mu}\phi^{,\nu}$ and the power-low potential $V(\phi) = V_0\phi^N$ is considered. An influence of nonzero scalar field potential on the cosmological dynamics of the model is studied. The methods of the theory of dynamical systems is used in order to analyze all possible asymptotic regimes.

Main equations

Let us consider the theory of gravity with the action

$$S = \int d^4x\sqrt{-g}\left\{\frac{R}{8\pi} - \left[g^{\mu\nu} + \kappa G^{\mu\nu}\right]\phi_{,\mu}\phi_{,\nu} - 2V(\phi)\right\}, \tag{1}$$

where $V(\phi)$ is the scalar field potential, $g_{\mu\nu}$ is the metric tensor, R is the scalar curvature, $G_{\mu\nu}$ is the Einstein tensor, and κ is the coupling parameter with the dimension of *length*^{2}.

In the spatially-flat Friedmann-Robertson-Walker cosmological model with the metric signature $(-+++)$ the action (1) yields the following field equations

$$3H^2 = 4\pi\dot{\phi}^2\left(1 - 9\kappa H^2\right) + 8\pi V(\phi), \tag{2}$$

$$2\dot{H} + 3H^2 = -4\pi\dot{\phi}^2\left[1 + \kappa\left(2\dot{H} + 3H^2 + 4H\ddot{\phi}\dot{\phi}^{-1}\right)\right] + 8\pi V(\phi), \tag{3}$$

$$(\ddot{\phi} + 3H\dot{\phi}) - 3\kappa(H^2\ddot{\phi} + 2H\dot{H}\dot{\phi} + 3H^3\dot{\phi}) = -V_\phi, \tag{4}$$

where a dot denotes the derivative with respect to time, $H(t) = \dot{a}(t)/a(t)$ is the Hubble parameter, $a(t)$ is the scale factor, $\phi(t)$ is a homogenous scalar field, and $V_\phi = dV/d\phi$.

We introduce the following set of dimensionless variables

$$x = \frac{8\pi\dot{\phi}^2}{6H^2(1+8\pi\kappa\dot{\phi}^2)}, \quad y = -\frac{8\pi\kappa\dot{\phi}^2}{2(1+8\pi\kappa\dot{\phi}^2)}, \quad z = \frac{8\pi V}{3H^2(1+8\pi\kappa\dot{\phi}^2)}, \quad v = \frac{\dot{\phi}}{\phi H} \tag{5}$$

and also the dimensionless parameter depending on the specific form of function $V(\phi)$:

$$\beta = \frac{\phi V_\phi}{V} \tag{6}$$

In this work we chose the power-low potential $V(\phi) = V_0\phi^N$ so that $\beta = \frac{NV_0\phi^N}{V_0\phi^N} = N = const.$

Generally, x characterizes the kinetic energy, and z characterizes the potential energy of the scalar field, while y is connected with the non-minimal kinetic coupling. Correspondingly, $z = 0$ if $V = 0$, and $y = 0$ if $\kappa = 0$.

Taking derivative of variables x, z, v (y is excluded by $x + y + z = 1$ which follows from (2)) with respect to $\ln a$ ($' := \frac{d}{d\ln a}$), we obtain:

$$\begin{aligned}
x' &= \tfrac{x}{5z^2+4x^2-9x(1-z)-11z+6} \\
&((Nvz(x+z-2) - 6(1-z)(x+z))(3-2x-2z)- \\
&-2Nvz(x+z-1) - 6(1-z)(2x+3z-3))), \\
\\
z' &= \tfrac{z}{5z^2+4x^2-9x(1-z)-11z+6} \\
&(-2Nvz(x+z-1) - 6(1-z)(2x+3z-3)+ \\
&+2(Nvz(x+z-2) - 6(1-z)(x+z))(1-x-z)) + Nvz, \\
\\
v' &= \tfrac{v}{5z^2+4x^2-9x(1-z)-11z+6} \\
&(\tfrac{1}{2}Nvz(x+z-2) - 3(1-z)(x+z)- \\
&-Nvz(x+z-1) - 3(1-z)(2x+3z-3)) - v.
\end{aligned} \tag{7}$$

Stationary points, their stability and the corresponding solutions

In this section we study the stationary points of the dynamical system (7) and perform the stability and asymptotic analysis of these points. To find a stationary point (x_0, z_0, v_0), we set $x_0' = z_0' = v_0' = 0$ in (7) and solve the resulting algebraic equations. Then, we investigate its stability with respect to small perturbations δx, δz, and δv around (x_0, z_0, v_0). Specifically, we substitute

$$x = x_0 + \delta x, \quad z = z_0 + \delta z, \quad v = v_0 + \delta v \tag{8}$$

into (7) and keep the terms up to the first order in δx, δz, δv. This leads to the system of first-order ordinary differential equations

$$\frac{d}{d(\ln a)} \begin{pmatrix} \delta x \\ \delta z \\ \delta v \end{pmatrix} = \mathcal{M} \begin{pmatrix} \delta x \\ \delta z \\ \delta v \end{pmatrix}, \tag{9}$$

where $\mathcal{M}$ is a 3×3 matrix which depends on (x_0, z_0, v_0). The stability of the stationary point (x_0, z_0, v_0) is determined by the corresponding eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ of $\mathcal{M}$. In particular, if the real parts of all eigenvalues are negative, the point is stable (the local sink); if all real parts are positive, the point is unstable being stable while integrating in the opposite time direction (the local source); if there are eigenvalues with different signs of their real parts, the point is the saddle point.

The results of this investigation are presented in the Table 1 for the case of $N \neq 2$. At $N = 2$ the eigenvalues of some points are equal to zero so that this case must be studied separately. It is necessary to stress that all the solutions found here were checked by the substitution of those in the initial system of equations (2)-(4).

№	**Coordinates of stationary points**	**Corresponding solution**	**Character of stability**
1	$x = 0$, $y = 1$, $z = 0$, $v = 0$	$a(t) = a_0\|t - t_0\|^{\frac{2}{3}}$, $\phi(t) = \phi_0 \pm \sqrt{-\frac{1}{12\pi\kappa}}(t - t_0)$, Exists for $\forall N, \kappa < 0$, $t \to t_0$	Unstable node
2	$x = \frac{1}{2}$, $y = -\frac{1}{2}$, $z = 1$, $v = 0$	$a(t) = a_0 e^{\pm\sqrt{\frac{1}{3\kappa}}(t-t_0)}$, $\|\phi(t)\|^{\frac{2}{2-N}} = \frac{\sqrt{\|V_0\|(2-N)}}{2}(t - t_0)$, Exists for $0 < N < 2, \kappa > 0$ $t \to \infty$	Complex type, **stability depends on the initial data**
3	$x = 1$, $y = 0$, $z = 0$, $v = 0$	$a(t) = a_0\|t - t_0\|^{\frac{1}{3}}$, $\phi(t) = \pm\sqrt{\frac{1}{12\pi}} \ln\|\frac{t-t_0}{t'-t'_0}\|$, Exists for $V(\phi) \equiv 0, \forall\kappa$, $t \to \infty$	Saddle
4	$x = \frac{3}{2}$, $y = -\frac{1}{2}$, $z = 0$, $v = -3$	$a(t) = a_0 e^{\pm\frac{1}{3\sqrt{\kappa}}(t-t_0)}$, $\phi(t) = \phi_0 e^{-\frac{1}{\sqrt{\kappa}}(t-t_0)}$, Exists for $0 < N < 2, \kappa > 0$, $\phi \to \infty$, $t \to -\infty$	Unstable node
5	$x = 0$, $y = -\frac{1}{2}$, $z = \frac{3}{2}$, $v = \frac{12N}{3N+2}$	$a(t) = a_0\|t - t_0\|^{\frac{3N+2}{3(2-N)}}$, $\phi(t) = \phi_0\|t - t_0\|^{\frac{4}{2-N}}$, Exists for $N > 2, \forall\kappa$, $t \to t_0$	**Stable node**

Table 1. Stationary point, the corresponding solutions, and their character of stability for the case of $N \neq 2$.

The numerical study of the initial system (2)-(4) for $0 < N < 2$ shows that the trajectories asymptotically go to either the solution 2 from the Table 1 or to the oscillations depending on the initial data (see Fig. 1.)

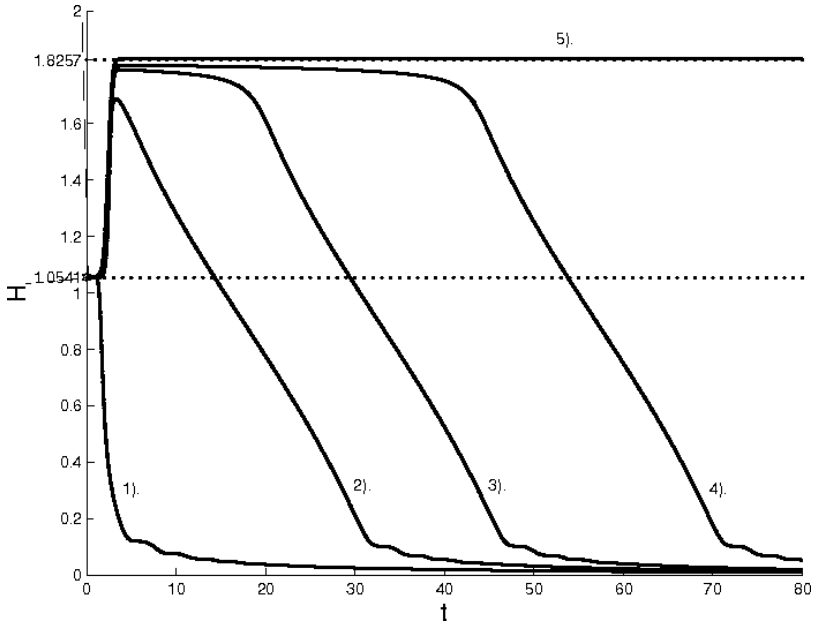

Figure 1: **Fig. 1.** The evolution of the Hubble parameter for the potential $V(\phi) = 0.1|\phi|^{\frac{3}{2}}$, the coupling constant $\kappa = 0.1$ and the following initial data: **1).** $\phi(0) = 32, \dot{\phi}(0) = -100$, **2).** $\phi(0) = 320, \dot{\phi}(0) = -1000$, **3).**$\phi(0) = 3200, \dot{\phi}(0) = -10000$,**4).**$\phi(0) = 32000, \dot{\phi}(0) = -100000$,**5).**$\phi(0) = 3200000, \dot{\phi}(0) = -11000000$. The lower and upper dotted line show the asymptotics: $1/(3\sqrt{\kappa}) \approx 1.0541$ and $1/\sqrt{3\kappa} \approx 1.8257$.

Conclusions

In this work the influence of the power-low potential $V(\phi) = V_0\phi^N$ $(N \neq 2)$ on the model of cosmological dynamics with the nonminimal kinetic coupling $\kappa G_{\mu\nu}\phi^{,\mu}\phi^{,\nu}$ is studied. Using the methods of the theory of dynamical systems two different **stable** asymptotic regimes are found:
(i) the power-low solution existing for $N > 2$, $t \to t_0$, and leading to the "Big Rip" singularity, and
(ii) the solution with the constant Hubble parameter $H = \pm\sqrt{\frac{1}{3\kappa}}$ and the power-low increase of the scalar field, which exists for $0 < N < 2$.

For the sloping potentials with $0 < N < 2$ the unstable exponential solution $H = \pm\sqrt{\frac{1}{9\kappa}}$ exists which earlier was found by Sushkov for the potential $V(\phi) = const$. Therefore for $N > 2$ this inflationary regime is destroyed.

Starting from one unstable exponential regime with $H = \pm\sqrt{1/9\kappa}$ the trajectories may go to another stable inflationary stage with $H = \pm\sqrt{1/3\kappa}$. It is an eternal inflation. However, our numerical integration of the initial system of equations (2)-(4) shows (see Fig. 1) that for a wide range of parameters κ and V_0 we have the possibility of a transition from the first inflation to oscillations. The range of the parameters for such scenario needs to be further investigated in more details. It is also necessary to study the case of $N = 2$ for which the eigenvalues of some stationary points are equal to zero.

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Nonlinear generalization of the Dirac equation

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Abstract: *We postulate a new nonlinear generalization of the Dirac equation for an electron. Basic properties of the new equation are considered.*

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Dirac equation for an electron. Let $\mathbb{R}^{1,3}$ be the Minkowski space with Cartesian coordinates x^μ, $\mu = 0,1,2,3$, with partial derivatives $\partial_\mu = \partial/\partial x^\mu$, and with a metric tensor given by the diagonal matrix $\eta = \mathrm{diag}(1,-1,-1,-1)$. Consider the Dirac equation for an electron

$$i\gamma^\mu(\partial_\mu\psi - ia_\mu\psi) - m\psi = 0, \tag{1}$$

where $\psi = \psi(x)$ is a Dirac spinor, a_μ are components of a covector potential of electromagnetic field, and γ^μ are are γ-matrices in the Dirac representation. We have

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu, \quad \mu \neq \nu, \quad (\gamma^0)^2 = \mathbf{1}, \quad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -\mathbf{1},$$

where $\mathbf{1}$ is the four dimensional identity matrix.

Let us recall basic properties of the Dirac equation.

1. The electric charge conservation law follows from the Dirac equation

$$\partial_\mu j^\mu = 0, \quad \text{where} \quad j^\mu = \bar\psi\gamma^\mu\psi = \psi^\dagger\gamma^0\gamma^\mu\psi,$$

where $\dagger$ is the operation of Hermitian conjugation.

2. The Dirac equation is covariant under Lorentz transformations of coordinates

$$x^\mu \to \acute{x}^\mu = p^\mu_\nu x^\nu, \quad \psi \to \acute\psi = S\psi,$$

where $P = \|p^\mu_\nu\| \in O(1,3)$, $S \in \mathrm{Pin}(1,3)$ and $S^{-1}\gamma^\mu S = p^\mu_\nu\gamma^\nu$ (definition of the Lie group $\mathrm{Pin}(1,3)$ see in [2]).

3. The Dirac equation is covariant under gauge transformations w.r.t. the U(1) gauge Lie group

$$\psi \to \hat\psi = e^{i\lambda}\psi, \quad a_\mu \to \hat a_\mu = a_\mu + \partial_\mu\lambda, \quad \lambda : \mathbb{R}^{1,3} \to \mathbb{R}.$$

4. The Dirac equation is related to the decomposition of the Klein-Gordon-Fock operator

$$(i\gamma^\mu\partial_\mu - m)(i\gamma^\nu\partial_\nu + m) = -(\partial_\mu\partial^\mu + m^2). \tag{2}$$

Generalized Dirac equation with nonlinearity. Let us take $\mathrm{I} = \gamma^0\gamma^1\gamma^2\gamma^3$. We see that $\mathrm{I}^2 = -\mathbf{1}$, $\mathrm{I}^\dagger = -\mathrm{I}$. Denote a subalgebra of matrix algebra

$$N = \{\alpha\mathbf{1} + \beta\mathrm{I} \in \mathrm{Mat}(4,\mathbb{C}) : \alpha,\beta \in \mathbb{R}\} \simeq \mathbb{C}.$$

$$Z = \alpha\mathbf{1} + \beta\mathrm{I} \leftrightarrow z = \alpha + i\beta \in \mathbb{C}.$$

Let $f = f(z)$ be a continuous function $f : \mathbb{C} \to \mathbb{C}$ and let $F = F(Z)$ be the corresponding function $F : N \to N$ such that $F(Z)|_{\mathbf{1}\to 1, \mathrm{I}\to i} = f(z)$.

Let us postulate the following equation, which depend on the function $F : N \to N$:

$$i\gamma^\mu(\partial_\mu\psi - ia_\mu\psi) - F(Z)\psi = 0, \tag{3}$$

where

$$Z = (\bar\psi\psi)1 - (\bar\psi\mathrm{I}\psi)\mathrm{I}$$

The first term in the equation (3) is equal to the first term in the Dirac equation for an electron (1). So, we say that the equation (3) is *a generalized Dirac equation (with a nonlinearity).*

Consider basic properties of the generalized Dirac equation (3).

1. The electric charge conservation law:

$$\partial_\mu j^\mu = 0, \quad \text{where} \quad j^\mu = \bar\psi\gamma^\mu\psi = \psi^\dagger\gamma^0\gamma^\mu\psi.$$

2. Lorentz invariance of the Dirac equation:

$$x^\mu \to \acute{x}^\mu = p^\mu_\nu x^\nu, \quad \psi \to \acute{\psi} = S\psi,$$

where $P = \|p^\mu_\nu\| \in \mathrm{SO}_+(1,3)$, $S \in \mathrm{Spin}_+(1,3)$ and $S^{-1}\gamma^\mu S = p^\mu_\nu\gamma^\nu$.

3. Gauge invariance w.r.t. U(1) gauge Lie group:

$$\psi \to \hat\psi = e^{i\lambda}\psi, \quad a_\mu \to \hat a_\mu = a_\mu + \partial_\mu\lambda, \quad \lambda : \mathbb{R}^{1,3} \to \mathbb{R}.$$

4. Decomposition of the second order operator:

$$(i\gamma^\mu\xi_\mu - F(Z))(i\gamma^\nu\xi_\nu + \overline{F(Z)}) = -(\xi_\mu\xi^\mu + |F(Z)|^2). \tag{4}$$

If $F(Z) = \sigma\mathbf{1} + \rho I$, where σ, ρ is functions $\mathbb{R}^{1,3} \to \mathbb{R}$, then $\overline{F(Z)} = \sigma\mathbf{1} - \rho I$, $|F(Z)|^2 = \sigma^2 + \rho^2$, and ξ^μ are commutative symbols.

We see two differences in the basic properties of equations (3) and (1).

- The equation (1) is invariant under Lorentz transformations of coordinates from the Lie group $O(1,3)$, but the equation (3) is invariant under Lorentz transformations of coordinates from the proper orthochronous Lorentz group $\mathrm{SO}_+(1,3)$.
- The decomposition (2) is, generally speaking, different to the decomposition (4).

Consider special cases of the generalized Dirac equation.

If we take $F(Z) \equiv m\mathbf{1}$ in (3), then we get the Dirac equation (1). That means the equation (3) is, in fact, a generalization of the equation (1).

Let us remind that the Dirac equation (1) can be derived from the Lagrangian

$$\mathcal{L} = \bar\psi i\gamma^\mu(\partial_\mu\psi - ia_\mu\psi) - m(\bar\psi\psi).$$

If we take $F(Z) \equiv Z = (\bar\psi\psi)1 - (\bar\psi\mathrm{I}\psi)\mathrm{I}$, then we get the equation

$$i\gamma^\mu(\partial_\mu\psi - ia_\mu\psi) - ((\bar\psi\psi)1 - (\bar\psi\mathrm{I}\psi)\mathrm{I})\psi = 0,$$

which can be derived from the Lagrangian

$$\mathcal{L} = \bar\psi i\gamma^\mu(\partial_\mu\psi - ia_\mu\psi) - \frac{1}{2}(\bar\psi\psi)^2 + \frac{1}{2}(\bar\psi\mathrm{I}\psi)^2.$$

Heisenberg's nonlinear field equation. One can see similarity between the generalized Dirac equation and the Heisenberg nonlinear field equation [1]

$$i\gamma^{\mu}(\partial_{\mu}\psi - ia_{\mu}\psi) - (\bar{\psi}\gamma_{\mu}\psi)\gamma^{\mu}\psi - (\bar{\psi}\gamma_{\mu}\mathrm{I}\psi)\gamma^{\mu}\mathrm{I}\psi = 0. \tag{5}$$

Heisenberg in [1] had made an attempt to create a unified field theory on the basis of his equation (5). So, it will be interesting to develop such a theory on the basis of new generalized Dirac equation (3).

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Monocosm: GR-Friendly Description of Quantum Systems

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Abstract: *I present an axiomatic modification of quantum mechanics with a* possible worlds *semantics capable of* predicting *essential "nonquantum" features of an observable universe model - the* topology *and* dimensionality *of spacetime, the* existence, *the* signature *and a* specific form *of a metric on it, and a set of naturally preferred directions (*vistas*) in spacetime unrelated to its metric properties. It is shown that the dynamics of the modifcation is represented by a hyper-Hamiltonian flow (superposition of three dependent Hamiltonian flows).*

Keywords: Quantum mechanics, general relativity, topos, logic, cosmology, quantum gravity.
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Introduction

I would like to give you a brief non-technical introduction to Monocosm formalism, a generalization of Quantum Mechanics, based on several relatively new results in mathematical physics. This generalization can be of interest, because it mimics the kinematic structure of General Relativity, and is capable of accounting for three basic,"non-quantum" features of the observable world: the observable dimensionality of spacetime, the existence of a metric on it, and the signature of this metric. Let us recall a few simple technical constructs from the foundations of Hilbert space formulation of Quantum Mechanics.

M-Bundle

Recall that any Hilbert space contains a natural principal bundle, with a complex or quaternionic ray as both the standard fiber, and the structure group of this bundle. The total space of this bundle is the punctured Hilbert manifold, the set of all nonzero vectors of the Hilbert space, with the natural manifold structure. The base space of the bundle is the projective Hilbert space. For a complex Hilbert space, this principal bundle looks like stacks of paper cups, because the standard fiber, the Lie group of nonzero complex numbers is two dimensional topological cylinder. For a quaternionic Hilbert space the fiber is the Lie group of nonzero quaternions, which is a four dimensional manifold, a topological product of a three-sphere and a real line. In both bundles the structure groups act on each fiber, producing a realization of the group in the set of states. They are both single component Lie groups, easy to work with, parallelizable and require a single coordinate patch. Thus, any quantum system is represented by a set of its states with an action of a very special group. We call this bundle the monocosm of the system, or just the M-bundle.

FLRW Metrics on Quaternions

The monocosm formalism is based in part on a recent discovery of a natural relativistic structure on the the standard fiber of the M-bundle. Indeed, the Lie group of nonzero quaternions carries a set of built-in, closed Friedman-Lemaitre-Robertson-Walker metrics, each generated by a foliation of hyper-surfaces, induced by the gradient of some real-valued function. Each foliation determines the scale factor of the metric it generates. Thus we have the following situation: a quantum system is represented by something that looks like a copy of spacetime, with something that looks like a preferred time. For reasons that will be given in the semantics of the theory, we call this preferred time the perceptible time of the observer. To get an idea of how this important solution of Einstein equations pops up in quaternions, we recall that quaternions constitute a unital algebra, in terminology of Mike Postnikov. Unital algebras are real finite dimensional associative linear algebras with identity. Such algebras have a rich structure. In particular, the set of its invertible elements constitute a Lie group. In case of quaternion algebra, it is the set of nonzero quaternions. Any real linear algebra is a real vector space equipped with a rank three tensor (structure tensor).

This tensor is once contravariant and twice covariant, that is, it accepts a single one-form argument and two vector argumets, outputting a real number. If you freeze the one-form argument, that is, keep it constant, but vary the vector arguments, the rank of the tensor is reduced, you have just twice covariant tensor. If it is symmetric, it is an inner product on the vector space of the algebra. This is called a principal inner product of the algebra, and the one-form that generates it is called the ether form. These two objects are intended to capture the notion of natural geometry encoded in algebraic structure. Of course, if there are too many principal inner products of all signatures, or none at all, the algebra does not really have the intrinsic geometry. Quaternions, however, have a distinct intrinsic geometry, because the only principal inner product is Minkowski inner product. Curiously enough, Euclidean inner product cannot be obtained this way, so in this sense, Euclidean geometry is not an intrinsic geometry of quaternions. Now, recall that the vector space of the algebra is the tangent space, at the identity, to the Lie group of its invertible elements. Every principal inner product is automatically extended along the left-invariant vector fields to cover the whole group, producing what we call the principal metric on the Lie group. For quaternions this metric is closed FLRW.

Kähler Structure

The Hermitian inner product on a Hilbert space is a complex or quaternion-valued function. Given a canonical basis of unit complex numbers and quaternions, it can be decomposed into real and imaginary parts. It is well known that such a decomposition results in a Kähler, or hyper-Kähler structure on the Hilbert manifold. The real part gives the Riemannian metric, and the imaginary part gives one or three symplectic forms, for the complex and quaternionic case, respectively. By the way, there is a common misconception regarding the notion of a canonical basis of unit quaternions, 1, i, j, k. The thing is, that in contrast with uniqueness of the complex canonical basis, 1, i, there are many such bases in the quaternionic case, so the canonical decomposition of a quaternion would be different for different bases. We should always keep this in mind, when working simultaneously with several distinct objects based on quaternions.

Geometric Quantum Mechanics

This natural Kähler structure is used in Geometric Quantum Mechanics, a beautiful endeavor seeking to describe Quantum Mechanics in terms classical Hamiltonian formalism, in the manner similar to Classical Mechanics. The metric part of the Kähler structure describes some probabilistic features of quantum systems, and the symplectic form governs the dynamics. Observables are real-valued functions on the Hilbert manifold, constructed out of expectation values of the corresponding Hermitian operators. Just like in Classical Mechanics, to each observable we associate, via the symplectic correspondence, its Hamiltonian vector field. Eigenstates are points in the Hilbert manifold, at which the Hamiltonian vector field becomes vertical with respect to the natural connection generated by the metric part of the Kähler structure. Eigenvalues are the corresponding values of the observable. Temporal evolution of a system is described by the Hamiltonian vector field of a preferred observable, the Hamiltonian. This procedure is called the first stage of geometrization of Quantum Mechanics. Since all the vectors in a ray represent the same state of a quantum system, there are many integral curves, corresponding to the same evolution of the system. To eliminate this redundancy, the Hamiltonian formalism in Geometric Quantum Mechanics is projected down to the base space of the complex M-bundle. The base space, the complex projective Hilbert space has a natural Kähler structure induced by the ambient Kähler structure of the Hilbert space. The Hilbert space becomes extraneous and usually not mentioned. This is the second stage of geometrization. We should point out that this procedure fails in quaternionic case since quaternionic projective Hilbert space does not admit a hyper-Kähler structure. But in the complex case it works beautifully, and, as well known, the dynamics of the complex Geometric Quantum Mechanics is essentially equivalent to the canonical Schroedinger evolution.

Hyper-Hamiltonian Dynamics

Hyper-Hamiltonian dynamics [SLIDE] extends the Canonical Hamiltonian Formalism of Classical Mechanics from symplectic domain into the domain of hyper-Kähler manifolds. Observables are quaternion regular functions, a generalization of the notion of a holomorphic function of complex analysis. To each observable we associate a vector field, called its hyper-field, which is now a superposition of three Hamiltonian vector fields corresponding to the three symplectic forms of the Kähler structure. Non-uniqueness of a canonical basis of unit

quaternions plays very important part here, since the three Hamiltonian vector fields change with a change of canonical basis, but their superposition remains constant. Temporal evolution of the system is just the evolution along the flow, generated by the hyper-field of a preferred observable, the Hamiltonian of the system.

Monocosm Formalism

Monocosm formalism deploys the first stage of geometrization of Quantum Mechanics. It utilizes the natural Hyper-Kähler structure of the Quaternionic Hilbert Space and Hyper-Hamiltonian dynamics. States are in one-to-one correspondence with points of the total space of the M-bundle, the punctured Hilbert manifold. In particular, different points in a quaternionic ray are considered representing different states of a quantum system. The rays themselves are referred to as Possible Worlds of the system. Each possible world is endowed with a FLRW metric, and also with a preferred foliation of three-surfaces, which is interpreted as perceptible time of the observer. Observables in Monocosm are quaternion regular functions. They are quaternion-valued functions constructed out of expectation values of anti-Hermitian operators, which allows us to take advantage of the full power of Hyper-Hamiltonian Dynamics. To each observable we associate its hyper-field, a superposition of three Hamiltonian vector fields. Just like in the geometric formulation of complex quantum mechanics, eigenstates of an observable are points at which its hyper-field becomes vertical with respect to the natural connection generated by the metric part of the hyper-Kähler structure. Again, eigenvalues are the corresponding values of the observable. The possible world that contains an eigenstate of a system is called an accessible world or eigenworld of the system. Time in quantum mechanics is just a parameter of the flow, generated by vector field of the Hamiltonian of the system. In monocosm, however, we have the perceptible time. Any state of the system singles out a three-dimensional hyper-surface in the possible world it belongs to. Just like in standard Quantum Mechanics, in monocosm formalism an ideal measurement is an ordered pair of states, the initial, and the final states of the measurement, the former being the state of the system at the time of the measurement, and the latter being an eigenstate of the system. The corresponding possible worlds are called the source and target world of the measurement, respectively. The target world is also referred to as the actual world of the measurement. The eigenstate also singles out one of the hyper-surfaces of perceptible time. We call this hyper-surface the hyper-surface of the present. So, due to richness of the internal structure of possible worlds, we have more data pertaining to the measurement. These data consist of three parts. In the first part we have the distance between the initial and final states, with respect to the Riemannian metric which is called the propensity metric in monocosm. The inverse of this distance is called the propensity of the measurement. It is related to the notion of probability: the larger the propensity, the more likely the measurement will result in a particular final state. The second part of the data is the eigenvalue of the final state, which is a quaternion. Given a basis on the quaternion algebra, the eigenvalue can be decomposed into a quadruple of real numbers. The third part of the data is the hyper-surface of the present. It is intended to carry the instantaneous gravitational information of the system. This is the basic technical setup of monocosm formalism.

Semantics

In most theoretical constructs that use the notion of an observer, for example in physics, observers are considered occupants of the universe, that is, they are objects existing, in some sense, among other objects of the universe. In monocosm, however, an observer is not an occupant, but an experient of the universe. Of course it does not mean that we deny the existence of the universe and all that stuff. We simply take the minimalistic approach to philosophy, and assume that our experiences definitely exist. We also realize that no matter what we perceive or conceive, the percepts and concepts are also our experiences. We conjecture that complex experiences are somehow composed out of simple ones, and that there are the simplest experiences, called elementary experiences. So, what we are trying to do in monocosm, is to describe some constructs of physics, like quantum systems, completely in terms of elementary experiences of an observer, considered as primitive, undefined entities. For each observer, there are some special subsets of elementary experiences called paradigms, which can be pictured as various alphabets of some language of thought of the observer. Intuitively, each letter of any of the alphabets is an elementary observation and also an elementary action of the observer. It turns out that paradigms are sufficiently structured objects. For example, superposition and composition of elementary experiences can be defined in every paradigm, and many of them can be approximated by real associative algebras with identity. The vector space of this algebra is intended to capture the notion of elementary observations, called reflexors, and the multiplicative monoid вЂ“ the notion of elementary actions, called effectors. We call

the vector space and the monoid, the sensory domain and the motor domain of the observer, respectively. To our observer-experient, the universe is just the totality of his experiences, more or less divided into subsets, subject to some rules of behavior. This is nicely modeled by the category of sets, the universe of classical mathematics, which is a topos, is special kind of a category, with the behavior of its objects and arrows governed by a well-defined logic. For the category of sets, the logic is two-valued Boolean, for other toposes it does not have to be. We assume that the observerвЂ™s contribution to the evolution of the universe is done through his motor domain which somehow acts on each system. Recall that a quantum system is a set with the left action of the group of nonzero quaternions. This is captured by the notion of an M-set, a set with left action of a monoid, and the universe is modeled by the category of M-sets which turns out to be a topos. The logic of this topos is called the operational logic of the observer. The main result of the observer theory is this: Given a universe of M-sets, where M is the motor domain of a paradigm, and assuming that the operational logic of the observer is two-valued Boolean, then the paradigm is isomorphic to the quaternion algebra.

Example: The Old World

Consider an example of a quantum system described in monocosm terminology. We call this system the Old World. The M-bundle of the Old World contains a single possible world, hence the source, target and actual worlds coincide for each measurement. The ambient time and the perceptible time also coincide. A temporal evolution of the system is orthogonal to the level sets of the perceptible time with respect to the propensity metric. Since the evolution vector field is vertical everywhere, every state of the system is an eigenstate. For each measurement the hyper-surface of the present is a three-sphere. In conventional terms this seems to correspond to the classical coarse-grained view of the universe - an observer at rest relative to CMB in the spacetime of a spatially closed FLRW cosmology.

Summary

Monocosm is a proper generalization of the canonical Quantum Mechanics with observable as anti-Hermitian, instead of Hermitian operators. Of course, in the complex case, either representation is completely valid, due to simple relationship between eigenvalues of Hermitian and anti-Hermitian operators. The only effect of using anti-Hermitian operators is that the imaginary unit migrates from the Schroedinger equation to eigenvalues. If we take this generalization seriously, it seems to tell us that Standard Quantum Mechanics is best suited for description of the Universe with two-dimensional spacetime, because two out of four dimensions are collapsed in each possible world. Monocosm also seems to tell us that our spacetime is a spatially closed FLRW universe. The currently accepted observation data suggest that if spatial curvature exists, it is quite small, and favor a flat FLRW universe, although the observations are indirect and based in part on Standard Quantum Mechanics, which we are trying to modify here. Still, in monocosm this geometry is an intricate part of the very foundations of the theory, and obtained with no reference to General Relativity or any other theories of gravity, which is, at the very least, interesting. Monocosm predicts some curious technical effects, a family of non-geodesic flows along left-invariant vector fields on spacetime. The most ambitious results of the formalism asserts that we could be using an incorrect mathematics in our attempts to study the universe, and offers us the mathematics of the topos of M-sets as the correct one, in its opinion, so to speak. One of the consequences is that using Lebesgue measure in computations over large distances, at the scale of galaxies, clusters and larger, may result in a systematic accumulating error. Perhaps this could have some applications to the problem of dark matter, although, again this is pure speculation. To summarize, monocosm formalism is a new, non-trivial generalization of quantum mechanics that incorporates some kinematic features of General Relativity.

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A simple self-organized structure in the example of a sequential growth dynamics of a Causal Set

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Abstract: *A simple algorithm of a sequential growth dynamics of an x-graph is considered. The x-graph is a directed acyclic dyadic graph. This model is a particular case of a causal set approach to quantum gravity. The set of vertices of an x-graph is a particular case of a causal set. The sequential growth of an x-graph is a non-deterministic addition of new vertices one by one. The algorithm to add vertices satisfies the causality principle. The considered particular example of the algorithm is based on some random walk on an x-graph. I proved that there is a nonzero probability to generate a self-organized connected repetitive structure. This structure consists of any initial connected finite x-graph and a set of infinite sequences of vertices that are connected by double edges.*

Keywords: random graph, directed graph, causal set, self-organization.
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Introduction

One of the approaches to quantum gravity is a causal set hypothesis. Consider two-dimensional Minkowski space (Fig. 1).

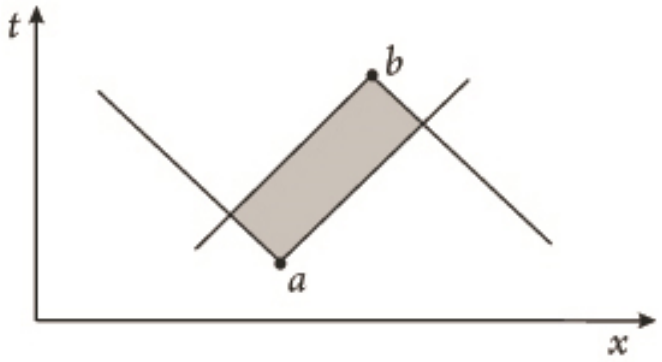

Figure 1: An Alexandov set in two-dimensional Minkowski space.

This is a partially ordered set of events. If two events a and b are causally connected, and a precedes b, we can consider an intersection of the future light cone of a and past light cone of b. This set is called an Alexandov set of a pair a and b. In Minkowski space, an Alexandov set of pairs of events is infinite or empty. According a causal set hypothesis we assume that an Alexandov set of pairs of events is finite or empty. Then the set of events in the universe is a partially ordered locally finite set (a causal set). A causal set approach to quantum gravity was introduced by G. 't Hooft [1] and J. Myrheim [2] in 1978 (see the review [3]).

A causal set can be represented as a directed acyclic graph. Vertices are images of elements of causal set, and directed paths are images of order relations. I investigate a simple particular case. This is an x-graph. An x-graph is a directed acyclic graph with the degree of any vertex no more than $(2, 2)$. This model was introduced by D. Finkelstein [4] in 1988.

In the causal set approach, there is only a causal set. All things in the universe are features of a causal set. There are two main problems. We must deduce spacetime as some continuous limit, and we must deduce matter. My talk covers the second problem. I suppose that the particles are some quasi-repetitive structures of a causal set. We can make such structures by hand. But quasi-repetitive structures must be self-organized structures as a consequence of dynamics. Let consider dynamics.

Any observer can only actually know a finite part of any process. Then we consider only finite x-graphs (Fig. 2). In a graph theory, by definition, an edge is a relation of two vertexes. Consequently some vertexes of finite x-graph have less than four incident edges. These vertexes have free valences instead the absent edges. These free valences are figured by external edges as external lines in Feynman diagrams. There are incoming and outgoing external edges. We can prove that the number of incoming external edges is equal to the number of outgoing external edges for any x-graph.

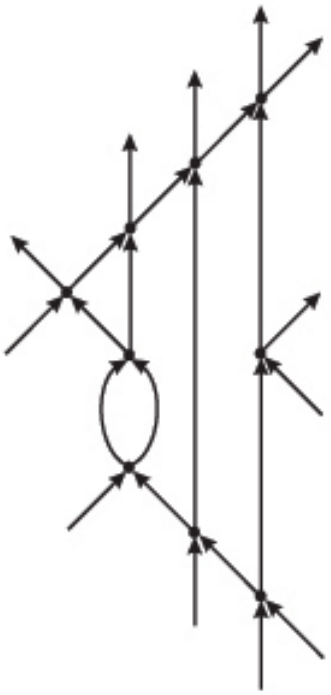

Figure 2: An example of x-graph.

A sequential growth dynamics

The task of any dynamics is to reconstruct the future or the past stages of processes. We can reconstruct an x-graph step by step. A minimal part is a vertex. We start from some given x-graph and add new vertexes one by one. This procedure is proposed in my papers [5,6] in 1998. Similar procedure and the term "a classical sequential growth dynamics" are proposed by D. P. Rideout and R. D. Sorkin [7] for other model of a causal set in 1999.

We can add a new vertex to external edges. This procedure is called an elementary extension. There are four types of elementary extensions (Fig. 3) [8]. In the figure the x-graph $\mathcal{G}$ is represented by a rectangle because it can have an arbitrary structure. There are two types of elementary extensions to outgoing external edges. This is a prediction of the future evolution of the process. First type is an addition of a new vertex to two outgoing external edges (Fig. 3a). Second type is an addition of a new vertex to one outgoing external edge (Fig. 3b). Similarly, there are two types of elementary extensions to incoming external edges. This is a reconstruction of the past evolution of the process. Third type is an addition of a new vertex to two incoming external edges (Fig. 3c). Fourth type is an addition of a new vertex to one incoming external edge (Fig. 3d). We can prove

that we can get every finite connected x- graph from a single vertex by a sequence of elementary extensions of these four types.

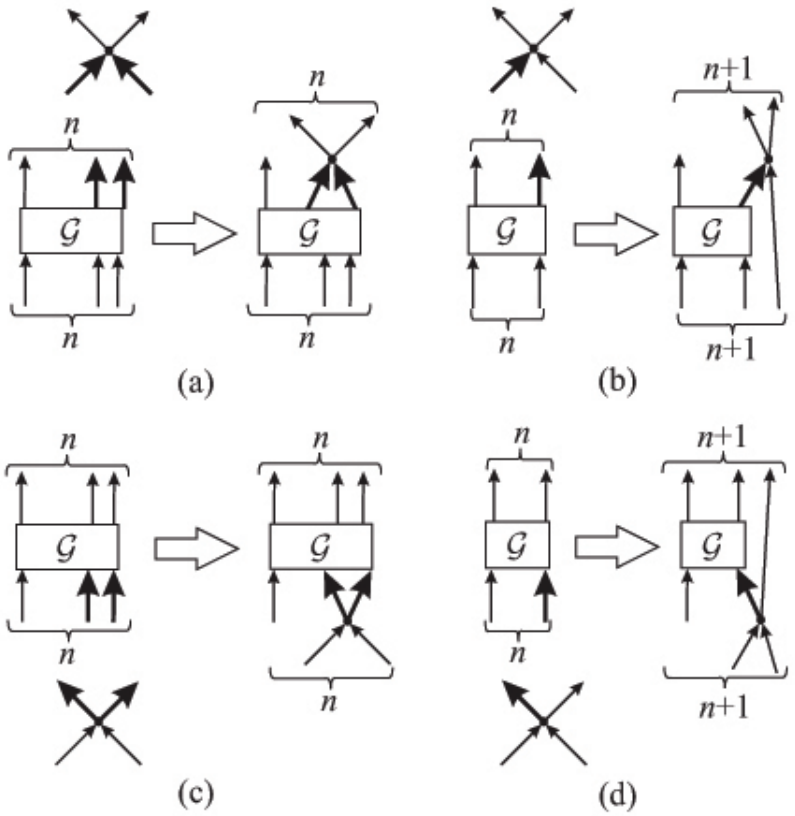

Figure 3: Four types of elementary extensions.

By assumption, the dynamics of this model is a non-deterministic dynamics. The algorithm to calculate probabilities of different variants of elementary extensions (or to choose the elementary extension) is a fundamental equation of motion for a causal set. I suppose that this algorithm is a very simple. I discovered one simple algorithm such that a sequential growth by this algorithm generates a simple quasi-regular infinite structure with nonzero probability. Let consider this algorithm.

An algorithm of growth

The algorithm to choose the elementary extension without the calculation of probabilities includes 3 steps. The first step is the choice of the elementary extension to the future or to the past. We assume the probability of this choice is 1/2. The second step is the choice of one external edge that takes part in the elementary extension. We assume the equiprobable choice. The probability of this choice is $1/n$, where n is the number of incoming or outgoing external edges in the x-graph. The third step is the final choice of the elementary extension. We use the following random walk at the x-graph (Fig. 4).

Let us enumerate the incoming or outgoing external edges by Greek and Latin indices respectively. Suppose we have chosen the outgoing external edge number i in the previous steps. Otherwise the procedure for incoming external edge is the same,if we change the direction of all edges of the x-graph. Start a random walk from the outgoing external edge number i in opposite direction to the direction of the edges (Fig. 4a). In each vertex (including the incident vertex of the edge number i), we choose the u-turn or the continuation of the walk with probability 1/2. If we have chosen the continuation of the walk, we choose one of two edges with probability 1/2 and go to the next vertex. If we have chosen the u-turn, we continue the walk in the direction of edges. We choose one of two edges with probability 1/2 and go to the next vertex. But we cannot choose the edge if

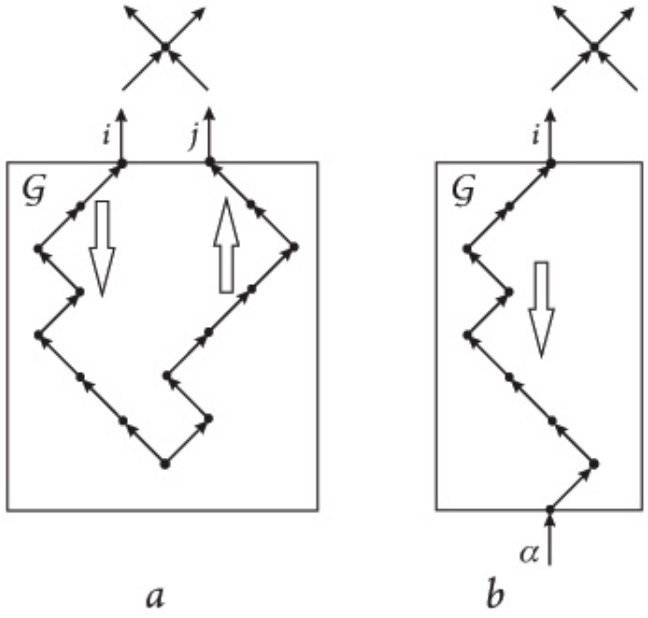

Figure 4: The random walk at the x-graph.

this edge was included in the path when we walked in opposite direction. In this case, we choose the other edge without an alternative. We end the walk in some outgoing external edge number j. This is the second external edge that takes part in the elementary extension. A new vertex is added to two outgoing external edges. We get the elementary extension of the first type. In this procedure, there is a special case (Fig. 4b). During the walk in the opposite direction to the edges we can go to the vertex that is incident to an incoming external edge number α. If we choose this edge number α, we cannot go to the next vertex because this vertex does not exist. In this case, we interrupt the walk and add a new vertex only to one outgoing external edge number i. We get the elementary extension of the second type.

In this model, causality is defined as the partial order of vertices. But the causality has a real dynamical meaning. The probability to add a new vertex to the future can depend only on the subgraph that precedes this vertex [7]. Similarly, the probability to add a new vertex to the past can depend only on the subgraph that follows this vertex. Only the normalization constant depends on the other parts of the x-graph. The considered algorithm satisfies the causality principle.

A simple repetitive infinite structure

This algorithm can generate a simple repetitive infinite structure with nonzero probability. Let begin from a single vertex number 0. In the first step, the probability to get two vertices that are connected by a double edge is $q_1 = 1/2$. Denote by N the number of steps during the growth. Denote by q_N the probability to get a sequence of $N + 1$ vertices connected by N double edges at the step number N. We have

$$q_{N+1} = (1 - 2^{-(N+1)})q_N. \tag{1}$$

This is iterative formula. We get an explicit equation.

$$q_{N+1} = (1-2^{-(N+1)})(1-2^{-N})\dots(1-2^{-2})\tfrac{1}{2} =$$

$$= \tfrac{1}{2} + \tfrac{-1}{2^2}\sum_{k_1=1}^{N} 2^{-k_1} + \tfrac{(-1)^2}{2^3}\sum_{k_1=2}^{N}(2^{-k_1}\sum_{k_2=1}^{k_1-1} 2^{-k_2})+$$

$$+\tfrac{(-1)^3}{2^4}\sum_{k_1=3}^{N}(2^{-k_1}\sum_{k_2=2}^{k_1-1}(2^{-k_2}\sum_{k_3=1}^{k_2-1} 2^{-k_3}))+ \qquad (2)$$

$$+\tfrac{(-1)^m}{2^{(m+1)}}\sum_{k_1=m}^{N}(2^{-k_1}\sum_{k_2=m-1}^{k_1-1}(2^{-k_2}\cdots\sum_{k_m=1}^{k_{m-1}-1} 2^{-k_m}))+$$

$$+(-1)^N 2^{\frac{-(N+1)(N+2)}{2}}.$$

Consider the lower bound of q_{N+1}. Cast out all positive terms except 1/2. Using,

$$\sum_{k_i=1}^{k_{i-1}-1} 2^{-k_i} = 1-2^{-(k_{i-1}-1)}, \qquad (3)$$

$$\sum_{k_i=m-i+1}^{k_{i-1}-1} 2^{-k_i} = \sum_{k_i=1}^{k_{i-1}-1} 2^{-k_i} - \sum_{k_i=1}^{m-i} 2^{-k_i} =$$
$$= 1-2^{-(k_{i-1}-1)} - (1-2^{-(m-i)}) < 2^{-(m-i)} \le 1, \qquad (4)$$

we get

$$q_{N+1} \ge \frac{1}{2} - \frac{1}{2^2}\sum_{k=0}^{(N-2)/2} 2^{-4k} \text{ if } N \text{ is even}, \qquad (5)$$

$$q_{N+1} \ge \frac{1}{2} - \frac{1}{2^2}\sum_{k=0}^{(N-1)/2} 2^{-4k} \text{ if } N \text{ is odd}. \qquad (6)$$

Consider the limit of infinite N.

$$q_\infty \ge \frac{7}{30}. \qquad (7)$$

We get the nonzero probability of generation of infinite sequence of vertices that are connected by double edges. Such sequence can be a part of a more complicated structure. Let begin from any finite x-graph. During sequential growth there is a nonzero probability a to get an x-graph such that all incoming or outgoing external edges form pairs that are incident to the same vertex (Fig. 5a). During sequential growth each this pair can form an infinite sequence of vertices that are connected by double edges with probability that no less than 7/30. Consequently there is nonzero probability that the sequential growth of any finite x-graph from some step is the growth of infinite sequences of vertices that are connected by double edges (Fig. 5b). This probability is no less than $a(7/30)n$, where n is the number of incoming or outgoing external edges.

Conclusions

We get the repetitive quasi-regular structure. Only a distribution of lengths of sequences is random at the each finite step. We can get rarely connected sequences of vertices by some modification of the algorithm. This first simple example shows that a non-deterministic sequential growth of a causal set can generate self-organized structures.

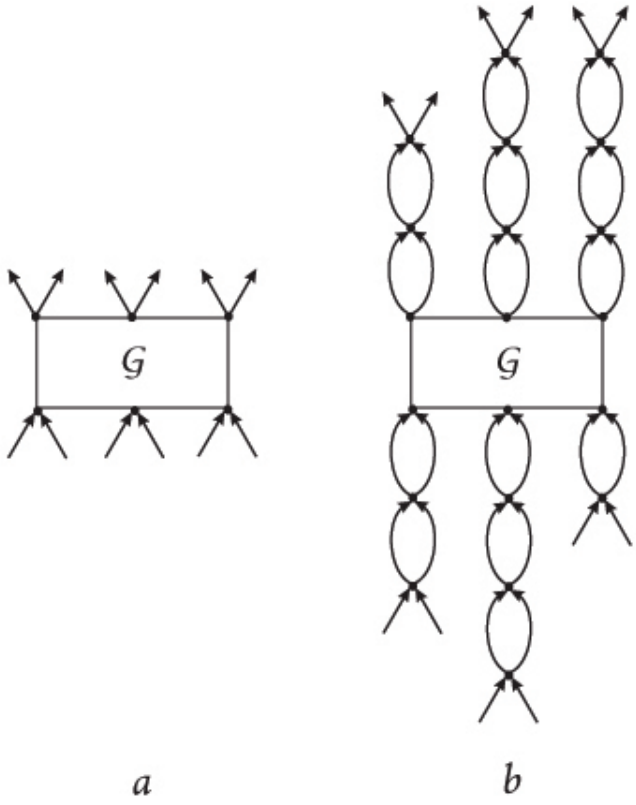

Figure 5: Two stages of sequential growth of structure.

I hope that:

- The origin of many problems in quantum theory is that quantum theory uses classical spacetime.
- The causal set hypothesis is a right step from classical spacetime to discrete pregeometry on a microscopic level.
- The particles are self-organized structures of a causal set.

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